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INDEX.

Lectures on the Theory of Reciprocants.

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[Reported by J. HAMMOND.]

LECTURE XI.

We may write for the Annihilator of an Invariant

$$\Omega = a_0\dot{a}_1 + 2a_1\dot{a}_2 + 3a_2\dot{a}_3 + \ldots . ja_{j-1}\dot{a}_j$$

and for its opposite

$$O = ja_1\dot{a}_0 + (j-1)\,a_2\dot{a}_1 + (j-2)\,a_3\dot{a}_2 + \ldots . + a_j\dot{a}_{j-1},$$

where the pointed letters $\dot{a}_0,\ \dot{a}_1,\ \dot{a}_2, \ldots .\ \dot{a}_j$ stand for the partial differential operators $\qquad \partial_{a_0},\ \partial_{a_1},\ \partial_{a_2}, \ldots .\ \partial_{a_j}$.

Suppose Ω and O to operate on any function $U(a_0,\ a_1,\ a_2, \ldots .\ a_j)$; then

$$\Omega O U = (\Omega . O + \Omega * O)\, U$$

and

$$O\Omega U = (O . \Omega + O * \Omega)\, U,$$

where the full stop between O and Ω signifies multiplication, and the asterisk operation on the unpointed letters only. Thus,

$$\Omega . O = O . \Omega,$$

and, consequently, $\qquad (\Omega O - O\Omega)\, U = (\Omega * O - O * \Omega)\, U.$

Now, $\quad \Omega * OU = \{1 . ja_0\dot{a}_0 + 2\,(j-1)\,a_1\dot{a}_1 + 3\,(j-2)\,a_2\dot{a}_2 + \ldots . + j . 1a_{j-1}\dot{a}_{j-1}\}\, U,$

and $\quad O * \Omega U = \{1 . ja_1\dot{a}_1 + 2\,(j-1)\,a_2\dot{a}_2 + \ldots . + (j-1)\,2a_{j-1}\dot{a}_{j-1} + j . 1a_j\dot{a}_j\}\, U,$

whence we readily obtain

$$(\Omega O - O\Omega)\, U = j\,(a_0\dot{a}_0 + a_1\dot{a}_1 + a_2\dot{a}_2 + \ldots . + a_j\dot{a}_j)\, U$$
$$-2\,(a_1\dot{a}_1 + 2a_2\dot{a}_2 + 3a_3\dot{a}_3 + \ldots . + ja_j\dot{a}_j)\, U.$$

Introducing the conditions of homogeneity and isobarism, viz.

$$(a_0\dot{a}_0 + a_1\dot{a}_1 + a_2\dot{a}_2 + \ldots . + a_j\dot{a}_j) \quad U = iU$$

and

$$(a_1\dot{a}_1 + 2a_2\dot{a}_2 + 3a_3\dot{a}_3 + \ldots . + ja_j\dot{a}_j)\, U = wU,$$

where i and w denote the degree and weight of U, supposed now to be a rational integral homogeneous and isobaric function (or, to avoid a tedious periphrasis, say a *gradient*), we see that if the complete type of the gradient U is $w; i, j$,

$$(\Omega O - O\Omega)\, U = (ij - 2w)\, U = \eta U,$$

where η is the excess.

Since the operation of O increases the weight of the operand by unity. but does not alter either its degree or its extent, it is clear that the type of $O^\theta U$ is $w + \theta; i, j$. The excess of $O^\theta U$ is therefore

$$ij - 2(w + \theta) = \eta - 2\theta,$$

and the theorem just proved shows that

$$(\Omega O - O\Omega)\, O^\theta U = (\eta - 2\theta)\, O^\theta U.$$

From this we pass on to prove that $\Omega O^q - O^q\Omega$, acting on any gradient as its objective, is equivalent to $q(\eta - q + 1)\, O^{q-1}$; *i. e.* when q is any positive integer we shall show that

$$(\Omega O^q - O^q\Omega)\, U = q(\eta - q + 1)\, O^{q-1} U.$$

The subsequent consideration of a special case of this formula, in which U is replaced by any invariant I, will enable us to prove that there can be no invariants for which the excess $ij - 2w$ is negative. Let

$$O^{q-\theta}\Omega O^\theta U = P_\theta U;$$

then $$O^{q-\theta-1}\Omega O^{\theta+1} U = P_{\theta+1} U,$$

and therefore $$(P_{\theta+1} - P_\theta)\, U = O^{q-\theta-1}(\Omega O - O\Omega)\, O^\theta U.$$

Substituting in this for

$$(\Omega O - O\Omega)\, O^\theta U \text{ its value } (\eta - 2\theta)\, O^\theta U,$$

we have $$(P_{\theta+1} - P_\theta)\, U = (\eta - 2\theta)\, O^{q-1} U.$$

Hence

$$\begin{aligned}(P_q - P_0)\, U &= \{(P_1 - P_0) + (P_2 - P_1) + (P_3 - P_2) + \ldots + (P_q - P_{q-1})\}\, U \\ &= \{\eta + (\eta - 2) + (\eta - 4) + \ldots + (\eta - 2q + 2)\}\, O^{q-1} U \\ &= q(\eta - q + 1)\, O^{q-1} U.\end{aligned}$$

But since $P_q = \Omega O^q$ and $P_0 = O^q\Omega$, this result may be written

$$(\Omega O^q - O^q\Omega)\, U = q(\eta - q + 1)\, O^{q-1} U.$$

If now $U = I$, an invariant, we have $\Omega U = 0$, and our formula becomes

$$\Omega O^q I = q(\eta - q + 1)\, O^{q-1} I.$$

Writing in succession $q = m, m - 1, \ldots 1$, we obtain

$$\begin{aligned} m(\eta - m + 1)\, O^{m-1} I &= \Omega O^m I \\ (m - 1)(\eta - m + 2)\, O^{m-2} &= \Omega O^{m-1} I \\ (m - 2)(\eta - m + 3)\, O^{m-3} I &= \Omega O^{m-2} I \\ \ldots\ldots &\ldots\ldots \\ 1 . \eta I &= \Omega O I. \end{aligned}$$

By assigning to m a sufficiently large value we are able to make $O^m I$ vanish as well as ΩI; for, the type of I being $w; i, j$, that of $O^m I$ is $w+m; i, j$. But it is evident that no *gradient* can have a greater weight than ij, the product of its degree and extent, for each term is a product of i letters none of them having a weight greater than j. If, then, we suppose that $m = ij - w + 1$, the weight of $O^m I$ is

$$w + m = ij + 1.$$

Therefore

$$O^m I = 0.$$

Again,

$$\eta - m + 1 = ij - 2w - (ij - w + 1) + 1 = -w.$$

If, then, η is negative, every term in the series

$$m(\eta - m + 1),\ (m-1)(\eta - m + 2), \ldots .\ 2(\eta - 1),\ 1.\eta$$

is negative and can never vanish. Hence we have successively

$$O^{m-1}I = 0,\ O^{m-2}I = 0, \ldots .\ I = 0;$$

i. e. when $ij - 2w < 0$ no invariant of the type $w; i, j$ exists.

Observe that the elenchus of the demonstration consists in the fact that the successive numerical factors $\eta - m + 1, \eta - m + 2, \eta - m + 3, \ldots . \eta$ are all non-zero on account of η being negative; but if η were positive we should eventually come to a factor $\eta - \mu$ which would be zero, and we could not conclude from $(\mu + 1)(\eta - \mu)\, O^\mu I$ being zero that $O^\mu I = 0$. Since $\eta - (m-1)$ passes from $\eta - (ij - w)$ to η, *i. e.* from $-w$ to η, it passes through zero when η is positive.

The second part of Cayley's completed theorem remains to be proved, viz. that when $ij - 2w \geqq 0$, the number of linearly independent invariants of the type $w; i, j$ is precisely equal to $\Delta(w; i, j)$; *i. e.* to $(w; i, j) - (w-1; i, j)$. I show this by proving that if $D(w; i, j)$ is the number in question, keeping i and j constant and taking $w \leqq \frac{ij}{2}$,

$$D(w; i, j) + D(w-1; i, j) + D(w-2; i, j) + \ldots . + D(0; i, j)$$

cannot be greater than

$$\Delta(w; i, j) + \Delta(w-1; i, j) + \Delta(w-2; i, j) + \ldots . + \Delta(0; i, j),$$

and consequently, since we know that no single $D(w; i, j)$ can possibly be less than the corresponding $\Delta(w; i, j)$, it follows that

$$D(w; i, j) + D(w-1; i, j) + D(w-2; i, j) + \ldots . + D(0; i, j)$$
$$= \Delta(w; i, j) + \Delta(w-1; i, j) + \Delta(w-2; i, j) + \ldots . + \Delta(0; i, j);$$

and, furthermore, that each

$$D(w; i, j) = \Delta(w; i, j).$$

For if any D were greater than its corresponding Δ, some other D would have to be less, which is impossible.

This principle of reasoning may be illustrated by imagining a row of ballot-boxes and supposing it to be ascertained that no single box contains fewer white balls than black ones. If, then, there are not more white than black balls altogether, the total number of whites must be the same as that of the blacks. And since there are just as many whites as blacks distributed among the ballot-boxes, the number of white and black balls must be the same in each box; for otherwise some box must contain fewer whites than blacks, which is contrary to the hypothesis.

Observe that the sum of the Δ's is $(w; i, j)$; for

$$(w;i,j)-(w-1;i,j)+(w-1;i,j)-(w-2;i,j)+\ldots.+(0;i,j)-(-1;i,j)$$
$$=(w;i,j)-(-1;i,j)$$

and
$$(-1;i,j)=0,$$

since there is no way of composing -1 with parts $0, 1, 2, \ldots. j$. Hence what I have to show is that

$$D(w;i,j)+D(w-1;i,j)+\ldots.+D(1;i,j)+D(0;i,j)=(w;i,j).$$

I want preliminarily to express $\Omega^q O^q I$ as a multiple of I.*

This can be done by a formula previously demonstrated, viz.

$$\Omega O^q I = q(\eta - q + 1)\, O^{q-1} I,$$

which gives

$$\Omega^2 O^q I = q(\eta - q + 1)\,\Omega O^{q-1} I = q(\eta - q + 1)(q-1)(\eta - q + 2)\, O^{q-2} I;$$

similarly

$$\Omega^3 O^q I = q(\eta - q + 1)(q-1)(\eta - q + 2)(q-2)(\eta - q + 3)\, O^{q-3} I;$$

and finally, changing the order of the numerical factors,

$$\Omega^q O^q I = 1.2.3\ldots. q\,\{\eta(\eta-1)(\eta-2)\ldots.(\eta - q + 1)\}\, I.$$

This shows that $\Omega^q O^q I$ and *à fortiori* $O^q I$ can never vanish unless $\eta - q + 1$ becomes negative.

Suppose now that I^q means an invariant of the type $w-q; i, j$; its excess is $ij - 2(w-q)$, and consequently $O^q I_q$ cannot vanish unless $ij - 2(w-q) - q + 1$ becomes negative, which is impossible. For

$ij - 2(w-q) - q + 1 = ij - 2w + q + 1$, and $ij - 2w \geqq 0$ by hypothesis.

By taking $O^q I_q$ as an *image*, so to say, of I_q we shall be able to obtain a limit to the number of I_q's by obtaining a limit to the number of their images. In fact, taking the *image* $O^q I_q$ of each of the $D(w-q; i, j)$ linearly independ-

* The result of operating on I with O and Ω each q times, the two operations following each other according to any law of distribution whatever, will always be a numerical multiple of I; but the value of this multiple will differ for different laws of distribution.

ent invariants of the type $w-q;\ i,\ j$ (this is what is meant by the I_q's) and giving q all possible values from 0 to w inclusive, the total number of these images is obviously

$$D(w;i,j)+D(w-1;i,j)+\ldots\ldots+D(0;i,j).$$

Each of them will be a gradient of the weight $w-q+q$ (*i. e.* of weight w), and will consist of terms of weight w, degree i, and extent j. The total number of such terms will be the number of ways of making up w with i of the numbers $0, 1, 2, 3, \ldots\ldots j$, or with the usual notation $(w;\ i,\ j)$. If, then, it can be shown that none of these forms are linearly connected, then, inasmuch as they are all functions of the same $(w;i,j)$ arguments, it will follow that their total number cannot exceed $(w;i,j)$. *I. e.* we shall have shown that

$$D(w;i,j)+D(w-1;i,j)+D(w-2;i,j)+\ldots\ldots+D(0;i,j)$$

cannot exceed

$$\Delta(w;i,j)+\Delta(w-1;i,j)+\Delta(w-2;i,j)+\ldots\ldots+\Delta(0;i,j),$$

and by the ballot-box principle, as already stated (inasmuch as no D is less than its corresponding Δ), it will follow that each D is the same as the corresponding Δ, and the theorem to be proved is established.

The proof of this independence is easy. For 1° suppose that there is any linear relation between the forms

$$O^qI_q,\ O^qI'_q,\ O^qI''_q,\ \ldots\ldots,$$

for each of which the value of q is the same. Denoting these forms by

$$P_q,\ P'_q,\ P''_q,$$

let the relation in question be

$$\lambda P_q+\lambda' P'_q+\lambda'' P''_q+\ldots\ldots=0.$$

Then
$$\lambda\Omega^qP_q+\lambda'\Omega^qP'_q+\lambda''\Omega^qP''_q+\ldots\ldots=0.$$

But each argument Ω^qP_q is of the form $\Omega^qO^qI_q$, and since this is equal to I_q multiplied by a number which does not vanish,* we have a linear relation between $I_q,\ I'_q,\ I''_q,\ \ldots\ldots$, viz.

$$\lambda I_q+\lambda' I'_q+\lambda'' I''_q+\ldots\ldots=0;$$

i. e. the I_q's would not be linearly independent, contrary to hypothesis. Thus the *images* $(O^qI_q,\ O^qI'_q,\ O^qI''_q\ \ldots\ldots)$ belonging to invariants of the same type $w-q$; i, j cannot be linearly connected.

2°. I say that the images of invariants of different types cannot be linearly connected. For let $q,\ q',\ q'',\ \ldots\ldots$ arranged in descending order of magnitude,

---

* In fact, remembering that the excess of the type $w-q;\ i,j$ is $ij-2(w-q)=\eta+2q$, we find

$$\Omega^qO^qI_q=1.2.3\ldots\ldots q\,\{(\eta+2q)(\eta+2q-1)\ldots\ldots(\eta+q+1)\}\,I_q,$$

in which both η and q are *positive* integers.

be the different values of q in the images supposed to be linearly related. The result of operating with Ω^q on any image of the form $O^{q'}I_{q'}$ is to bring it to the form $\Omega^{q-q'}\Omega^{q'}O^{q'}I_{q'}$, which is a multiple of $\Omega^{q-q'}I_{q'}$, and therefore vanishes. But Ω^q, acting on any of the images O^qI_q, $O^qI'_q$, ..., will, as we have seen, bring back the multiple of I_q; thus the operation of Ω^q on the supposed relation will give a linear equation connecting I_q, I'_q, I''_q, ... and for the same reason as before this is impossible. Hence there can be no linear relation whatever between the images of the invariants whose types extend from $w : i . j$ to $0 : i . j$, and the number of these images will accordingly be not greater than $(w : i . j)$, as was to be proved.

It is well worthy of notice that $D(w; i, j)$ may be zero, but obviously cannot be negative, as it denotes a number of things which may have any value from zero upwards. Hence follows a remarkable theorem in the pure theory of partitions which it would be extremely difficult to prove from first principles, viz. that the difference between the two partition numbers

$$(w; i, j) - (w-1; i, j)$$

can never be negative when $ij - 2w => 0$. It may be zero, but cannot be less than zero. This explains what I said about the hyperbolic paraboloid $ij - 2w = 0$, where i, j, w are treated as co-ordinates of a point in space. We might call the value of $(w;\ i,\ j) - (w-1; i, j)$ the density of any point $i.\ j.\ w$, and the theorem may then be expressed by saying that at points within or upon the hyperbolic paraboloid the density can never be negative; for points outside this surface it can never be positive.

As regards the analogous formula in the Theory of Reciprocants

$$(w; i, j) - (w-1;\ i+1, j),$$

we do not know that any algebraical surface can be constructed which will enable us to discriminate between the cases in which this difference, say $E(w:\ i.\ j)$, is positive or negative. Should such a surface exist, its equation must contain w in a higher degree than the first. Supposing that the above formula represents the actual number of reciprocants, it will follow (and this is confirmed by experience) that there can be no reciprocants to a type of negative excess. For

$$\begin{aligned} &(w; i, j) - (w-1; i+1, j) \\ &= (w; i, j) - (w-1; i, j) - [(w-1; i+1, j) - (w-1; i, j)] \\ &= (w; i, j) - (w-1; i, j) - (w-i-2; i+1, j-1). \end{aligned}$$

But if $ij - 2w$ is negative, $(w; i, j) - (w-1: i, j)$ is zero or negative. Hence $(w; i, j) - (w-1;\ i+1, j)$ is non-positive.

For *satisfied* invariants (those ordinarily so called) $w = \frac{ij}{2}$, and the formula for their number becomes $\left(\frac{ij}{2}; i, j\right) - \left(\frac{ij}{2} - 1; i, j\right)$.

As these form a well-defined class apart, it would have seemed very natural to begin with them in endeavoring to establish the theorem, reserving the theory of unsatisfied invariants (sources of covariants) for future consideration. But to all appearance it would have been very difficult, if not impossible, to have succeeded in dealing with them alone.

This is another example of the law in Heuristic that the whole is easier of deglutition than its part.

LECTURE XII.

Before proceeding further with the development of the pure analytical theory of reciprocants, it may be useful to point out some instances of its relations and applications to geometrical questions.

Using $y_1, y_2, y_3, \ldots . y_n$ to denote the successive derivatives of y with respect to x,* let the complete primitive of the differential equation

$$F(x, y, y_1, y_2, \ldots . y_n) = 0$$

be

$$\phi(x, y, \lambda, \mu, \nu, \ldots .) = 0.$$

We can in general so determine the n constants $\lambda, \mu, \nu, \ldots .$ that the curve ϕ may pass through n given points, and if we take these to be consecutive points on the curve

$$\Phi(x, y) = 0,$$

ϕ and Φ will have a contact of the $(n-1)^{\text{th}}$ order at a given point of Φ. In order that the curves may have a contact of the n^{th} order at a point whose abscissa is x, the ordinates of Φ and ϕ at that point and their 1^{st}, 2^{d}, ... n^{th} derivatives with respect to x must be the same for both curves. But at every point of ϕ its differential equation

$$F(x, y, y_1, y_2, \ldots . y_n) = 0$$

has to be satisfied, and therefore the $x, y, y_1, y_2, \ldots . y_n$ of any point on Φ, at which contact of the n^{th} order with ϕ is possible, must also satisfy the same equation.

---

* In future $y_1, y_2, y_3, \ldots . y_n$ will always have this meaning, the derivatives of x with respect to y will be denoted by x_1; $x_2, x_3, \ldots .$, and whenever the letters $t, a, b, c, \ldots .$ are used they will stand for $y_1, \frac{y_2}{1.2}, \frac{y_3}{1.2.3}, \frac{y_4}{1.2.3.4}, \ldots .$ respectively.

Now, suppose that for x and y we substitute given functions of them, X and Y; the curves ϕ and Φ become

$$\phi(X,\ Y,\ \lambda,\ \mu,\ \nu,\ \ldots.) = 0 \text{ and } \Phi(X,\ Y) = 0.$$

Contact of the n^{th} order with the transformed ϕ will therefore be possible at any point of the transformed Φ for which

$$F(X,\ Y,\ Y_1,\ Y_2, \ldots.\ Y_n) = 0,$$

where $Y_1,\ Y_2,\ Y_3, \ldots.\ Y_n$ are the derivatives of Y with respect to X.

But, unless the function F and the substitutions $X = f_1(x,\ y),\ Y = f_2(x,\ y)$ are so related that the transformed differential equation

$$F(X,\ Y,\ Y_1,\ Y_2, \ldots.\ Y_n) = 0$$

is identical with the untransformed one, the property marked by the contact of the transformed curves will not be identical with that marked by the contact of the untransformed ones.

For example, let $F = y_2$; then the relation between $\phi \equiv y + \lambda x + \mu = 0$ (the complete primitive of $y_2 = 0$) and an arbitrary curve Φ is that the constants λ and μ may be so chosen that the line $y + \lambda x + \mu = 0$ may have a contact of the second order at any point of Φ for which $y_2 = 0$; and the property marked is an inflexion on Φ. But if we make the substitution $X = x^2,\ Y = y^2$, so that the differential equation $y_2 = 0$ is transformed into $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$ and its complete primitive into $y^2 + \lambda x^2 + \mu = 0$, it will still be possible so to choose λ and μ that $y^2 + \lambda x^2 + \mu = 0$ may have a contact of the second order at any point of an arbitrary curve for which $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$, but the property marked, instead of being an inflexion, will be a *contact of the second order with a conic having a pair of conjugate diameters coincident with the co-ordinate axes.*

This property remains unaltered when the co-ordinate axes are interchanged, and therefore the differential equation $\left(\frac{d}{dx^2}\right)^2 y^2 = 0$ will be identical with $\left(\frac{d}{dy^2}\right)^2 x^2 = 0$, in which the variables x and y have changed places. The identity of the two differential equations is easily verified, for

$$\left(\frac{d}{dx^2}\right)^2 y^2 = \frac{1}{2x} \cdot \frac{d}{dx}\left(\frac{y}{x} \cdot \frac{dy}{dx}\right) = \frac{1}{2x} \cdot \left\{\frac{y}{x} \cdot \frac{d^2y}{dx^2} + \frac{1}{x}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x^2} \cdot \frac{dy}{dx}\right\}$$

$$= \frac{1}{2x^3}(xyy_2 + xy_1^2 - yy_1);$$

so that the differential equation may be written

$$xyy_2 + xy_1^2 - yy_1 = 0.$$

Interchanging x and y in this, we have

$$yxx_2 + yx_1^2 - xx_1 = 0,$$

in which, if we write $x_1 = \frac{dx}{dy} = \frac{1}{y_1}$, and $x_2 = \frac{d^2x}{dy^2} = -\frac{y_2}{y_1^3}$, it follows immediately that

$$yxx_2 + yx_1^2 - xx_1 = -\frac{1}{y_1^3}(xyy_2 + xy_1^2 - yy_1),$$

and the identity in question is established.

Such a form as the above, which merely acquires an extraneous factor when the variables are interchanged, might be called a reciprocant, if it were not convenient to restrict the use of the word to forms in which the variables x and y do not appear explicitly. With this limitation, the geometrical property indicated by the evanescence of a reciprocant will be independent of the position of the origin, but not in general independent of the directions of the co-ordinate axes. Thus, we may prove that the equation

$$2y_1y_3 - 3y_2^2 = 0$$

indicates the possibility of 4-point contact with a hyperbola whose asymptotes are *parallel to the co-ordinate axes.* To do this it is sufficient to show that its complete primitive is the equation to such a hyperbola.

Writing the equation in the form

$$\frac{y_3}{y_2} = \frac{3}{2} \cdot \frac{y_2}{y_1},$$

we see that its first integral is

$$\log y_2 = \frac{3}{2} \log y_1 + \text{const.};$$

or, when prepared for a second integration,

$$-\frac{1}{2} \cdot y_1^{-\frac{3}{2}} y_2 = \lambda.$$

Hence

$$y_1^{-\frac{1}{2}} = \lambda x + \mu,$$

$$y_1 = (\lambda x + \mu)^{-2},$$

and finally we obtain the complete primitive

$$\lambda(\nu - y) = (\lambda x + \mu)^{-1},$$

which proves the proposition.

With the notation previously explained, in which $y_1 = t$, $y_2 = 2a$, $y_3 = 6b$, the differential equation is $bt - a^2 = 0$. We have therefore proved that at all

points of a general curve for which the Schwarzian $(bt - a^2)$ vanishes, 4-point contact with a hyperbola whose asymptotes are parallel to the co-ordinate axes is possible.

We now consider the important case in which the conditioning differential equation remains unchanged when the axes are orthogonally transformed, and is therefore found by equating to zero an orthogonal reciprocant. The simplest example of this class of equations is that which marks the points of maximum or minimum curvature on a curve. Since these points are points of 4-point contact with a circle, the conditioning differential equation will be that of the circle

$$(x+\lambda)^2 + (y+\mu)^2 + \nu = 0.$$

Differentiating this three times in succession, we have

$$x + \lambda + (y+\mu)t = 0,$$
$$1 + t^2 + 2a(y+\mu) = 0,$$
$$at + b(y+\mu) = 0.$$

Eliminating μ from the last two of these equations, y will disappear at the same time, and the condition for points of maximum or minimum curvature is found to be

$$2a^2t - b(1+t^2) = 0.$$

In Salmon's Higher Plane Curves (2d edition, p. 357) the "aberrancy of curvature" is given by the formula

$$\tan\delta = y_1 - \frac{(1+y_1^2)\,y_3}{3y_2^2} = t - \frac{(1+t^2)\,b}{2a^2}.$$

The above differential equation is therefore equivalent to $\delta = 0$.

If we differentiate the radius of curvature $\rho = \dfrac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \dfrac{(1+t^2)^{\frac{3}{2}}}{2a}$, we find

$$\frac{d\rho}{dx} = \frac{6a^2t(1+t^2)^{\frac{1}{2}} - 3b(1+t^2)^{\frac{3}{2}}}{2a^2} = 3(1+t^2)^{\frac{1}{2}}\tan\delta = 3\tan\delta\,.\frac{ds}{dx}.$$

Hence it follows that

$$\tan\delta = \frac{1}{3}\,.\,\frac{d\rho}{ds}.$$

The conditioning equation for points at which $\dfrac{d\rho}{ds}$ or $\tan\delta$ is a maximum or minimum is $\dfrac{d^2\rho}{ds^2} = 0$; or the same condition may be expressed by $\dfrac{d\tan\delta}{dx} = 0$.

Now,
$$\frac{d\tan\delta}{dx} = \frac{d}{dx}\left\{t - \frac{b(1+t^2)}{2a^2}\right\} = 2a - \frac{2c(1+t^2)}{a^2} - \frac{2abt}{a^2} + \frac{3b^2(1+t^2)}{a^3}$$

is an orthogonal reciprocant, for it can be expressed in terms of legitimate

combinations of $1+t^2$, which is an orthogonal reciprocant of even character, with the three orthogonal reciprocants of odd character,

$$a,\ b(1+t^2)-2a^2t,\ c(1+t^2)-5abt+5a^3.$$

In fact, the above expression for $\frac{d\tan\delta}{dx}$, when multiplied by a^3 to clear of fractions, becomes,

$$\begin{aligned}&2a^4-2a^2bt+3b^2(1+t^2)-2ac(1+t^2)\\ &=\frac{3}{1+t^2}\{b(1+t^2)-2a^2t\}^2+\frac{12a^4}{1+t^2}-2a\{c(1+t^2)-5abt+5a^3\},\end{aligned}$$

where the right-hand side is a linear function of orthogonal reciprocants of the same (even) character, so that the combination is legitimate.

Quantities such as $\rho,\ \frac{d\rho}{ds},\ \frac{d^2\rho}{ds^2},\ \ldots\ldots$, or $\rho,\ \frac{d\rho}{d\phi},\ \frac{d^2\rho}{d\phi^2},\ \ldots\ldots$, where $d\phi$ is the angle subtended by the arc ds at the centre of curvature, have values independent of the particular position of the co-ordinate axes (supposed rectangular), and consequently these values, expressed in terms of $t,\ a,\ b,\ c,\ \ldots\ldots$ will be absolute orthogonal reciprocants. A differential equation expressing the condition that any one of these quantities vanishes, or that any one of them has a maximum or minimum value, will also be independent of the position of the rectangular axes, and must therefore be expressible in the form of an orthogonal reciprocant equated to zero.

Mr. Hammond remarks that, since the radii of curvature at corresponding points of a curve and its evolute are ρ and $\frac{d\rho}{d\phi}$, the radius of curvature of its n^{th} evolute is $\frac{d^n\rho}{d\phi^n}$. The radius of curvature of the n^{th} evolute of any n^{th} involute of a circle is constant, and, consequently, the differential equation of an n^{th} involute to a circle is

$$\frac{d^{n+1}\rho}{d\phi^{n+1}}=0.$$

Writing this in the form

$$\left(\frac{1+t^2}{a}\cdot\frac{d}{dx}\right)^{n+1}\cdot\frac{(1+t^2)^{\frac{3}{2}}}{a}=0,$$

to which it is easily reduced, since

$$\frac{d}{d\phi}=\rho\cdot\frac{d}{ds}=\frac{\rho}{(1+t^2)^{\frac{1}{2}}}\cdot\frac{d}{dx}=\frac{(1+t^2)}{2a}\cdot\frac{d}{dx},$$

we see by what precedes that the left-hand member of the differential equation is an orthogonal reciprocant.

As an example of the class of singularities which next presents itself for consideration, let us find the differential condition which holds at points of contact of the fourth order with a common parabola. This condition is expressible by the differential equation whose complete primitive is

$$(y + \varkappa x)^2 + 2\lambda x + 2\mu y + \nu = 0.$$

Differentiating three times in succession, we obtain

$$(y + \varkappa x)(t + \varkappa) + \lambda + \mu t = 0,$$
$$2a(y + \varkappa x + \mu) + (t + \varkappa)^2 = 0,$$
$$b(y + \varkappa x + \mu) + a(t + \varkappa) = 0.$$

The arbitrary constants ν and λ do not appear in the last two of these equations, from which, if we eliminate μ, the variables x and y disappear at the same time, and we find $$2a^2 - b(t + \varkappa) = 0.$$

A final differentiation and elimination give

$$10ab - 4c(t + \varkappa) = 0,$$
$$4ac - 5b^2 = 0.$$

Points of 5-point contact with a parabola are therefore indicated by the evanescence of the pure reciprocant $4ac - 5b^2$. And in general the differential equation $R = 0$, where R is any pure reciprocant, indicates a property of a curve which may be called a descriptive singularity, since it is totally unaffected by the arbitrary choice of any two lines on the plane for the axes of co-ordinates. For it was proved in Lecture IX of the present course that if i be the degree and μ the characteristic of R, the substitution of $ly + mx + n$ for x and $l'y + m'x + n'$ for y changes R into $(l'm - lm')^i(lt + m)^{-\mu}R$, so that the differential equation $R = 0$ and the geometrical property corresponding to it are left unchanged by the substitution.

Six-point contact with a cubical parabola is another example of a descriptive singularity. Its defining differential equation may be written in any of the following forms:

$$45y_2^3y_5^2 - 450y_2^2y_3y_4y_5 + 192y_2^2y_4^3 + 400y_2y_3^3y_5 + 165y_2y_3^2y_4^2 - 400y_3^4y_4 = 0,$$
$$125a^3d^2 - 750a^2bcd + 256a^2c^3 + 500ab^3d + 165ab^2c^2 - 300b^4c = 0,$$
$$5(9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3)^2 + 64(3y_2y_4 - 5y_3^2)^3 = 0,$$
$$125(a^2d - 3abc + 2b^3)^2 + 4(4ac - 5b^2)^3 = 0;$$

or, if we make $a^2d - 3abc + 2b^3 = A$ and $ac - \frac{5}{4}b^2 = M$, the equation may be put in the form $$\left(\frac{A}{16}\right)^2 + \left(\frac{M}{5}\right)^3 = 0.$$

In the theory of Binary Forms, when the numerical parameter x in

$$(a^2d - 3abc + 2b^3)^2 + x(ac - b^2)^3$$

is so chosen that the highest powers of b cancel each other, the form divides by a^2 and gives the Discriminant of the Cubic

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2.$$

In the parallel theory of Reciprocants the form

$$125A^2 + 256M^3$$

is divisible by a (instead of by a^2), giving

$$125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c,$$

which may be called the Quasi-Discriminant.

A complete discussion of the differential equation

$$A^2 + xM^3 = 0$$

is reserved for the next ensuing lecture, in the course of which it will appear that the Quasi-Discriminant equated to zero is the differential equation of the cubical parabola.

LECTURE XIII.

We may integrate the general homogeneous equation in reciprocants extending to d, inclusive, as follows:

Calling $\quad ac - \frac{5}{4}b^2 = M$ and $a^2d - 3abc + 2b^3 = A$,

the equation in question will be of the form

$$A^2 + xM^3 = 0.$$

But if we write $\quad \beta = \Lambda\alpha^{\lambda}$,

where β, α are general linear functions of the co-ordinates, say

$$y + mx + n,\ y + m'x + n',$$

we may eliminate the five constants m, n, m', n', Λ, and the result will evidently be a pure reciprocant extending to d, inclusive, and, being homogeneous and isobaric, can only be of the form

$$A^2 + xM^3 = 0,$$

so that it remains only to determine x in terms of λ, or, which is the same thing, λ in terms of x.

The solution $\beta = \Lambda\alpha^\lambda$ implies $\alpha = \Lambda^{-\frac{1}{\lambda}}\beta^{\frac{1}{\lambda}}$. Hence the equation between M and A must be of the form

$$\theta\{(\lambda + p)(p\lambda + 1)\}^i M^3 + \{(\lambda + q)(q\lambda + 1)\}^j A^2 = 0,$$

where θ is a constant, for otherwise there would be more than one general solution to it. It only remains then to determine the values of p, q, θ, i, j, which may be affected by considering the particular solution $y = x^\lambda$.

When $\lambda = 2$, M and A both vanish, and if $\lambda = 2 + \varepsilon$, where ε is an infinitesimal, M and A will each be of the same order as ε (that the first power of ε does not vanish in M or A may be easily verified). Hence $2 + q + \varepsilon$ is of the order ε, and therefore $q = -2$ and $j = 1$.

When $\lambda = -1 + \varepsilon$, M remains finite and A is of the order ε. Hence $p = 1$ and $i = 1$. Thus, the equation is

$$\theta(\lambda + 1)^2 M^3 + (\lambda - 2)(2\lambda - 1)A^2 = 0.$$

To find θ, let $\lambda = 3$ and $y = x^3$; then

$$a = 3x,\ b = 1,\ c = 0,\ d = 0,\ M = -\frac{5}{4},\ A = 2,$$

so that $$-\theta.\frac{5^3}{4} + 5.4 = 0,\ \theta = \frac{16}{25},$$

and finally $$16(\lambda + 1)^2 M^3 + 25(2\lambda^2 - 5\lambda + 2)A^2 = 0$$

has for its integral $$\beta = \Lambda\alpha^\lambda.$$

If $\lambda = \infty$, we may make

$$y = \left(1 + \frac{x}{\lambda}\right)^{\lambda l} = e^{lx},$$

and, consequently, $\beta = e^{l\alpha}$, which contains five independent arbitrary constants, will be the general integral.

For a parallel method of deducing the Integral of $A^8 + x\Delta^3 = 0$, where Δ (our future $AC - B^2$) is the projective reciprocant whose letters go up to f, see Halphen's Thèse sur les Invariants Différentiels, Paris, 1878.

Mr. Hammond has succeeded in deducing the equation between A and M from the primitive $\beta = \Lambda\alpha^\lambda$ by direct elimination, as shown in what follows. Possibly he, or some other algebraist, may eventually succeed in the more difficult task of obtaining the Differential Equation to $\gamma = \beta^\lambda\alpha^{1-\lambda}$ (*i. e.* the linear relation between A^8 and Δ^3) by some similar direct process.

Differentiating the equation $\beta\alpha^{-\lambda}=\lambda$ three times in succession, and observing that, since $\alpha=y+mx+n$ and $\beta=y+m'x+n'$,

$$\alpha''=\beta''=\frac{d^2y}{dx^2}=y_2,$$

we have
$$\alpha\beta'-\lambda\alpha'\beta=0,$$
$$y_2(\alpha-\lambda\beta)+(1-\lambda)\alpha'\beta'=0,$$
$$y_3(\alpha-\lambda\beta)+y_2\{(2-\lambda)\alpha'+(1-2\lambda)\beta'\}=0.$$

From the last two of these three equations we obtain, by eliminating $(\alpha-\lambda\beta)$,

$$y_3(1-\lambda)\alpha'\beta'-y_2^2\{(2-\lambda)\alpha'+(1-2\lambda)\beta'\}=0;$$

or, writing

$$y_2=2a,\ y_3=6b,\ 2-\lambda=3q^2,\ 1-2\lambda=-3r^2,\ 1-\lambda=q^2-r^2,$$

and dividing by $\alpha'\beta'$, the equation assumes the form

$$\frac{b}{2a^2}(q^2-r^2)=\frac{q^2}{\beta'}-\frac{r^2}{\alpha'}.$$

Differentiating again, remembering that $\alpha''=\beta''=2a$, and $\frac{da}{dx}=3b$, $\frac{db}{dx}=4c$, we find

$$\frac{4ac-6b^2}{4a^4}(q^2-r^2)=-\frac{q^2}{\beta'^2}+\frac{r^2}{\alpha'^2}.$$

The elimination of β' between this and the equation immediately preceding it gives

$$\frac{4ac-6b^2}{4a^4}(q^2-r^2)q^2+\left\{\frac{b}{2a^2}(q^2-r^2)+\frac{r^2}{\alpha'}\right\}^2-\frac{q^2r^2}{\alpha'^2}=0.$$

Writing in this $4ac-5b^2=4M$, we obtain by an easy reduction

$$4q^2Ma'^2=r^2\{2a^2-ba'\}^2,$$

and, taking the square root of each side,

$$\alpha'(2q\surd M+rb)-2a^2r=0.$$

A final differentiation gives

$$\alpha'\left(\frac{qM'}{\surd M}+4cr\right)+2a(2q\surd M-5br)=0.$$

Finally, eliminating α', we obtain

$$(2q\surd M+rb)(2q\surd M-5rb)+ar\left(4cr+\frac{qM'}{\surd M}\right)=0.$$

Hence
$$4Mq^2+qr\left(\frac{aM'}{\surd M}-8b\surd M\right)+r^2(4ac-5b^2)=0;$$

or,
$$4(q^2+r^2)M^{\frac{3}{2}}+qr(aM'-8bM)=0.$$

Now,
$$M'=\frac{dM}{dx}=\frac{d}{dx}\left(ac-\frac{5b^2}{4}\right)=5ad-7bc,$$

and, consequently,

$$aM' - 8bM = a(5ad - 7bc) - b(8ac - 10b^2) = 5(a^2d - 3abc + 2b^3) = 5A;$$

so that we may write

$$4(q^3 + r^3)M^{\frac{3}{2}} = -qr(aM' - 8bM) = -5qrA:$$

or,

$$16(q^3 + r^3)^2M^3 - 25q^2r^2A^2 = 0.$$

where

$$3q^3 = 2 - \lambda \text{ and } -3r^3 = 1 - 2\lambda.$$

Replacing q^3 and r^3 by their expressions in terms of λ, the differential equation becomes

$$16(\lambda + 1)^2M^3 + 25(2\lambda^2 - 5\lambda + 2)A^2 = 0.$$

Some special cases may be noticed.

When $\lambda = 2$ or $\frac{1}{2}$, the equation reduces to $M = 0$, which is the differential equation of the common parabola previously obtained.

When $\lambda = 3$ or $\frac{1}{3}$, we obtain $256M^3 + 125A^2 = 0$ for the equation of the cubical parabola, where the expression on the left-hand side is the Quasi-Discriminant.

When $\lambda = -1$, we find $A = 0$ for the differential equation of the general conic.

When λ is an imaginary cube root of negative unity, so that $\lambda^2 - \lambda + 1 = 0$, we have

$$(\lambda + 1)^2 + (2\lambda^2 - 5\lambda + 2) = 0,$$

and the differential equation becomes

$$16M^3 - 25A^2 = 0.$$

We shall subsequently avail ourselves of this result in finding the complete primitive of the Halphenian Δ.

In the case where λ is infinite, from the complete primitive $\beta = e^{l\alpha}$ we first eliminate the exponential function and afterwards the arbitrary constant l.

Thus we find

$$\beta' = l\alpha'\beta \text{ and } \frac{y_2}{\beta'} = \frac{y_2}{\alpha'} + \frac{\beta'}{\beta};$$

or,

$$y_2\beta(\alpha' - \beta') - \alpha'\beta'^2 = 0.$$

Hence

$$y_3\beta(\alpha' - \beta') - y_2\beta'(\alpha' + 2\beta') = 0.$$

The elimination of β gives

$$y_3\alpha'\beta' - y_2^2(\alpha' + 2\beta') = 0;$$

or,

$$\frac{3b}{2a^2} = \frac{1}{\beta'} + \frac{2}{\alpha'}.$$

Comparing this with the equation previously obtained,

$$\frac{b}{2a^2}(q^2 - r^2) = \frac{q^2}{\beta'} - \frac{r^2}{\alpha'},$$

we see that $q^2 = 1$ and $r^2 = -2$. Substituting these values in the differential equation

$$16(q^2 + r^2)^2 M^3 - 25 q^2 r^2 A^2 = 0,$$

it becomes

$$8M^3 + 25A^2 = 0,$$

which is the differential equation corresponding to the complete primitive $\beta = e^{i\alpha}$.

We shall hereafter consider in detail the theory of that special class of pure reciprocants (M. Halphen's Differential Invariants) which retain their form when any homographic substitution is impressed on the variables; *i. e.* when, instead of x and y, we write

$$\frac{lx + my + n}{l''x + m''y + n''} \text{ and } \frac{l'x + m'y + n'}{l''x + m''y + n''}.$$

Since perspective projection is the geometrical equivalent of homographic substitution, it follows from the definition of Differential Invariants that they are connected with the properties and relations of curves which remain unaffected by perspective projection. For this reason Differential Invariants are sometimes called Projective Reciprocants. Two reciprocants with which we are familiar belong to this important class. One of them, y_2 or a, vanishes at points of inflexion on the curve $y = f(x)$; the other,

$$9y_2^2 y_5 - 45 y_2 y_3 y_4 + 40 y_3^3, \text{ or } a^2 d - 3abc + 2b^3,$$

which, for reasons given below, we shall call the Mongian, vanishes at sextactic points; *i. e.* at points where a conic can be drawn having 6-point contact with the given curve.

To illustrate the distinction between a projective and a merely descriptive singularity, consider for an instant the pure reciprocant $4ac - 5b^2$, which, as we have seen, vanishes at all points of a general curve where 5-point contact with a parabola is possible. Now, 5-point contact with a parabola is a descriptive but not a projective singularity; after projection the parabola becomes a general conic, and 5-point contact with it becomes 5-point contact with a general conic, which is not a singularity at all. But inflexions and sextactic points are indelible by projection, and thus belong to the class of projective singularities.

The differential equation to a conic was originally obtained by Monge in the form

$$9y_2^2 y_5 - 45 y_2 y_3 y_4 + 40 y_3^3 = 0$$

(see Monge, Sur les Équations différentielles des Courbes du Second Degré. Corresp. sur l'École Polytech. Paris, II, 1809–13, pp. 51–54, and Bulletin de la Soc. Philom., Paris, 1810, pp. 87, 88). At the end of the first chapter of his Differential Equations, Boole mentions this form of equation as due to Monge, but without any reference, and adds the remark: "But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms." The theory of Reciprocants, however, furnishes both a simple interpretation of the Mongian equation and an obvious method of integrating it.

To see that the differential equation of a conic is satisfied at the Sextactic points of a given curve, we have only to remember that at such points the derivatives of y with respect to x, up to the fifth order, inclusive, are the same for the given curve as for a conic.

We proceed to show how the Mongian may be integrated. Writing in the above equation

$$y_2 = 2a,\ y_3 = 2.3b,\ y_4 = 2.3.4c,\ y_5 = 2.3.4.5d,$$

it becomes

$$a^2d - 3abc + 2b^3 = 0,$$

where it can hardly fail to be noticed that the left-hand member of the equation is an ordinary Invariant as well as a Reciprocant. It will be proved hereafter that all Differential Invariants possess this double nature.

Now, if $\mu = 3i + w$, where i is the degree and w the weight of any pure reciprocant R, the ordinary theory of eduction shows that

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \frac{a\frac{dR}{dx} - \mu bR}{a^{\frac{\mu}{3}+1}}$$

is another pure reciprocant.

When we consider the letters $a, b, c, \ldots.$ in any invariant I to mean $\frac{y_2}{2}, \frac{y_3}{2.3}, \frac{y_4}{2.3.4}, \ldots.$ the parallel theory of generation for Invariants gives the corresponding theorem that if $\nu = 3i + 2w$, where i is the degree and w the weight of I,

$$\frac{d}{dx}\left(\frac{I}{a^{\frac{\nu}{3}}}\right) = \frac{a\frac{dI}{dx} - \nu bI}{a^{\frac{\nu}{3}+1}}$$

is also an invariant.

A strict proof of this theorem will subsequently be given. For present

purposes it is sufficient to notice the easily verified special cases of the two theorems

$$\frac{d}{dx}\left(\frac{4ac-5b^2}{a^{\frac{8}{3}}}\right)=\frac{20\,(a^2d-3abc+2b^3)}{a^{\frac{11}{3}}}$$

and

$$\frac{d}{dx}\left(\frac{ac-b^2}{a^{\frac{10}{3}}}\right)=\frac{5\,(a^2d-3abc+2b^3)}{a^{\frac{13}{3}}}.$$

It follows as an immediate consequence that the equation

$$a^2d-3abc+2b^3=0$$

admits of the two first integrals

$$a^{-\frac{8}{3}}\,(4ac-5b^2)=\text{const.}$$

and

$$a^{-\frac{10}{3}}\,(ac-b^2)=\text{const.}$$

Now,

$$a^{-\frac{8}{3}}\,(4ac-5b^2)=\frac{d}{dx}\,(a^{-\frac{5}{3}}b)=-\frac{1}{2}\,\frac{d^2}{dx^2}\,(a^{-\frac{2}{3}});$$

so that the Mongian equation is equivalent to $\frac{d^3}{dx^3}\,(a^{-\frac{2}{3}})=0$, or to $\frac{d^3}{dx^3}\,(y_2^{-\frac{2}{3}})=0$. We thus obtain an integral of the form

$$y_2^{-\frac{2}{3}}=l+2mx+nx^2,$$

from which the complete primitive may be found by two easy integrations. Thus,

$$y_1+p=\int\frac{dx}{(l+2mx+nx^2)^{\frac{3}{2}}}=\frac{m+nx}{(ln-m^2)(l+2mx+nx^2)^{\frac{1}{2}}}$$

gives

$$y+px+q=\frac{1}{ln-m^2}\,(l+2mx+nx^2)^{\frac{1}{2}},$$

which is the equation of a general conic.

By first interchanging the variables x, y in the Mongian equation (whose form remains unaltered by this interchange, since $a^2d-3abc+2b^3$ is a reciprocant) and then integrating three times with respect to x, we should find another integral of the form

$$x_2^{-\frac{2}{3}}=l'+2m'y+n'y^2.$$

The solution may be completed by two integrations, as in the former method.

Mr. Hammond remarks that $\frac{2\,(ac-b^2)}{a^{\frac{10}{3}}}=\frac{d^2}{dt^2}\,(a^{\frac{2}{3}})$, where $t=y_1$. For, since

$$\frac{d}{dt}=\frac{dx}{dt}\cdot\frac{d}{dx}=\frac{1}{2a}\cdot\frac{d}{dx},$$

we have

$$\frac{d}{dt}\,(a^{\frac{2}{3}})=\frac{1}{2a}\cdot\frac{2}{3}\cdot a^{-\frac{1}{3}}\,.\,3b=\frac{b}{a^{\frac{4}{3}}},$$

and, consequently,

$$\frac{d^2}{dt^2}\,(a^{\frac{2}{3}})=\frac{1}{2a}\cdot\frac{d}{dx}\,(a^{-\frac{4}{3}}b)=2a^{-\frac{10}{3}}\,(ac-b^2).$$

Hence the integral $a^{-\frac{10}{3}}(ac - b^2) = \text{const.}$ previously obtained for the Mongian is equivalent to $\frac{d^2}{dt^2}(a^{\frac{2}{3}}) = \text{constant}$; *i. e.* to $\frac{d^2}{dy_1^2}(y_2^{\frac{2}{3}}) = \text{const.}$ Thus we have another integral of the form

$$y_2^{\frac{2}{3}} = \lambda + 2\mu y_1 + \nu y_1^2,$$

from which it is also easy to pass to the complete primitive.

I add a few general remarks relating to the subject-matter of this and the preceding lecture. Instead of the cumbrous terms Projective Reciprocants or Differential Invariants, it may be better to use the single word Principiants to denominate that crowning class or order of Reciprocants which remain, to a factor près, unaltered for any homographic substitutions impressed on the variables. This is the *species princeps*. If we go back to the *species infima*, we see the beginning of life in the subject. In general Reciprocants, all that is affirmed is that there exist forms-functions of the derivatives of y in regard to x which (to a factor près) remain unaltered when the variables x and y are interchanged, so that $f(y_1, y_2, y_3, \ldots.)$ becomes $\phi(x_1, x_2, x_3, \ldots.)$. The function ϕ only differs from f by the acquisition of an extraneous factor $(-)^\kappa y_1^\mu$; *i. e.*

$$f(y_1, y_2, y_3, \ldots.) = (-)^\kappa y_1^\mu f(x_1, x_2, x_3, \ldots.).$$

A particular species of these general (mixed) reciprocants arises when $f(y_1, y_2, y_3, \ldots.)$, differentiated in regard to y_1, gives a reciprocant. These are Orthogonal Reciprocants, and in them we see the first dawn of free continuous motion as distinguished from mere displacement (or mere interchange of axes). Orthogonal Reciprocants, when x, y are rectangular co-ordinates, remain unaltered (save as to a factor) when the orthogonal axes are moved continuously. A quarter of a revolution of course will reverse their original positions, so that we see the condition of mutual displacement is fulfilled. Thirdly, Reciprocants into whose form the first derivative y_1 does not enter are called Pure. Their form is invariable when the axes (now taken generally) undergo separate displacement (instead of turning round together) in a plane. Here there is a further development, so to say, of life in the subject.

Finally, in Principiants, a particular species of Pure Reciprocants, the invariance remains good, not merely for any position of the axes of reference, but for any homographic deformation of the plane in which they lie, so that the evanescence of a Principiant corresponds to some property of a curve not only

intrinsic but indelible by projection, as *ex. gr.* an inflexion, or a double point, or a sextactic point, and so on.

It is clear from this review that the Theory as we have given it goes to the root of the subject, and that the word Reciprocant is rightly chosen as conveying the notion of a property which is common to the entire continuous series of forms bearing that name. All the links of this connected chain are thus comprehended under the general name of Reciprocants.

LECTURE XIV.

The remaining lectures of the course will be devoted to the theory of Pure and Projective Reciprocants. I shall first treat of the existence and properties of the Protomorphs of Invariants and Reciprocants, using the latter system of protomorphs to obtain all the fundamental forms of Reciprocants in the letters a, b, c, d, e. I shall then pass on to the theory of Projective Reciprocants, or Principiants, with its applications contained in M. Halphen's Thèse pour obtenir le grade de docteur ès sciences (Paris, Gauthier Villars, 1878). It will be seen that M. Halphen's very ingenious methods become greatly simplified when his results are read by the light of an important discovery in the theory of Principiants recently made by myself and Mr. Hammond working conjointly, arising out of a theorem put forward by one of my hearers. This theorem, on examination, we found was necessarily erroneous and would fail at the very first step of its application. But although the proposition stated was wrong, it contained an Idea which survives and may be incorporated in a valid and extremely important theorem, which I will endeavor to explain.

A Principiant, besides being an Invariant in the original letters $a, b, c, d, \ldots.$ is also an Invariant in the letters $a, A, B, C, D, \ldots.$ where each capital letter is itself a Reciprocant; and, conversely, every invariant in the capital letters $A, B, C, D, \ldots.$ is a Principiant. The invariants in the capital letters form a system of protomorphs for Principiants, so that every Principiant is either some such invariant simply, or a rational integral function of such invariants divided by some power of a. Thus, for example, it will be proved that the Cubic Criterium (*i. e.* the Principiant which gives, when equated

to zero, the differential equation of a cubic curve) may be expressed as the quotient of

$$\frac{9}{64}A^5 + \frac{5}{4}A(A^2D - 3ABC + 2B^3) - (ACE - AD^2 - B^2E + 2BCD - C^3)$$

by the fifth power of a.

The proof of this theorem is based upon the fact that we can form a series of terms beginning with the Mongian (viz. $a^2d - 3abc + 2b^3$), say $A, B, C, D, \ldots$ such that

$$\Omega A = 0,$$
$$\Omega B = A \times \frac{a}{2},$$
$$\Omega C = 2B \times \frac{a}{2},$$
$$\Omega D = 3C \times \frac{a}{2},$$
$$\cdots\cdots\cdots\cdots$$

where
$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots,$$

coupled with the fact that every Principiant must be a function of the letters in such series and the small a.

Each consequent of the series $A, B, C, D, \ldots$ is, so to say, an Invariant relative to its antecedent; it becomes an actual Invariant when its antecedent vanishes.

In the theorem as originally proposed, each letter of the series was derived by the operation of an eductive generator upon the one which precedes. In the true theorem the scale of relation is between three and not two consecutive terms. Calling the letters $u_0, u_1, u_2, \ldots u_i$, we have

$$(i+7)u_{i+2} - Gu_{i+1} + (i+1)Mu_i = 0,$$

where G is the ordinary eductive generator,

$$4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \ldots,$$

M is the first pure reciprocant after the monomial a, viz. $M = ac - \frac{5}{4}b^2$, $u_0 = A = a^2d - 3abc + 2b^3$, and $6u_1 = GA$.

But although, as I have said, the theorem in the form proposed was absolutely erroneous, its proposer has rendered an invaluable service to the theory by the mere suggestion of what turns out to be true, viz. that every Principiant is an Invariant in regard to a known series of Reciprocants considered as simple elements.

To this theorem there is a correlative one, for it will be shown that there exists a series of invariants $A_0, A_1, A_2, \ldots$, the first term of which, A_0, is the same as the Mongian A, each of the other terms of the series being a Reciprocant relative to the one that precedes it. In fact, we have

$$VA_0 = 0,$$
$$VA_1 = -a^2A_0,$$
$$VA_2 = -2a^2A_1,$$
$$\ldots\ldots\ldots\ldots$$
$$VA_n = -na^2A_{n-1},$$

where
$$V = 4\left(\frac{a^2}{2}\right)\partial_b + 5ab\partial_c + 6\left(ac + \frac{b^2}{2}\right)\partial_d + \ldots,$$

and, as a consequence, every Principiant will be an Invariant in respect to these Invariants and the first small letter a.

Thus, speaking symbolically, we have not only

$$P = R + I$$

(a logical equation meaning that P has the same qualities as both R and I, or that a Principiant is both a Reciprocant and an Invariant), but also

$$P = IR \text{ and } P = II,$$

meaning that a Principiant is an Invariant of Reciprocantive elements, and an Invariant whose elements are themselves Invariants.

I may add that the invariantive elements $A_0, A_1, A_2, A_3, \ldots$ are defined by the equations

$$A_0 = A,$$
$$A_1 = B - \frac{b}{2}A,$$
$$A_2 = C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A,$$
$$A_3 = D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2 B - \left(\frac{b}{2}\right)^3 A,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

so that any invariant in the reciprocantive elements $A, B, C, D, \ldots$ is equal to the corresponding invariant in $A_0, A_1, A_2, A_3, \ldots$ Thus,

$$A = A_0,$$
$$AC - B^2 = A_0A_2 - A_1^2,$$
$$A^2D - 3ABC + 2B^3 = A_0^2A_3 - 3A_0A_1A_2 + 2A_1^3,$$
$$AE - 4BD + 3C^2 = A_0A_4 - 4A_1A_3 + 3A_2^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

M. Halphen appears not to have noticed the Principiant $AE - 4BD + 3C^2$, which presents itself naturally when the theory is viewed from our present ground of vantage, but A, $AC - B_2$ and $A^2D - 3ABC + 2B^3$ occur in his Thèse in connection with the curve

$$\alpha = \beta^\lambda \gamma^{1-\lambda},$$

in which α, β, γ are any linear functions of x, y, 1.

When $\lambda = -1$ the differential equation of this curve (the conic $\alpha\beta = \gamma^2$) is $A = 0$, but it is

$$AC - B^2 = 0$$

when λ is a cube root of negative unity, and

$$A^2D - 3ABC + 2B^3 = 0$$

when λ has an arbitrary value.

Before making out an exhaustive table of all the irreducible forms of pure reciprocants in the letters a, b, c, d, e similar to, but not identical with, the corresponding table for invariants, it seems to me desirable to say something of Protomorphs in general; and this will be better understood if we devote a short space to the protomorphs of Invariants. The simplest forms of these are the following well-known ones of alternately the second and third degrees:

$$P_2 = ac - b^2,$$
$$P_3 = a^2d - 3abc + 2b^3,$$
$$P_4 = ae - 4bd + 3c^2,$$
$$P_5 = a^2f - 5abe + 2acd + 8b^2d - 6bc^2,$$
$$P_6 = ag - 6bf + 15ce - 10d^2,$$
$$P_7 = a^2h - 7abg + 9acf - 5ade + 12b^2f - 30bce + 20bd^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

The quadratic Protomorphs $P_2, P_4, P_6, \ldots$, are absolutely unique, for the number of invariants of the type $j;\ 2, j$ is $(j;\ 2, j) - (j-1;\ 2, j) = 1$ if j is even, and $= 0$ if j is odd. Their form is so well known that there is no need to dilate upon it here.

The cubic ones $P_3, P_5, P_7, \ldots$, may be derived from the quadratic ones by means of Cayley's generators, given early in the course, viz.:

$$P = (ac - b^2)\partial_b + (ad - bc)\partial_c + (ae - bd)\partial_d + \ldots,$$
$$Q = (ac - 2b^2)\partial_b + 2(ad - 2bc)\partial_c + 3(ae - 2bd)\partial_d + \ldots$$

Let us first use the P generator

$$P(ac - b^2) = a(ad - bc) - 2b(ac - b^2) = a^2d - 3abc + 2b^3,$$
$$P(ae - 4bd + 3c^2) = a(af - be) - 4b(ae - bd) + 6c(ad - bc) - 4d(ac - b^2)$$
$$= a^2f - 5abe + 2acd + 8b^2d - 6bc^2.$$

Similarly, we find

$$P(ag-6bf+15ce-10d^2)=a^2h-7abg+9acf-5ade+12b^2f-30bce+20bd^2,$$

and so on.

Let I be any invariant whatever of the type $w; i, j$ (satisfied or unsatisfied); then using the original forms of the generators P and Q as given by Cayley (see Lect. IV), we have

$$PI=a(b\partial_a+c\partial_b+d\partial_c+\ldots)I-ibI,$$
$$QI=a(c\partial_b+2d\partial_c+3e\partial_d+\ldots)I-2wbI,$$

and, consequently,

$$(jP-Q)I=a\{jb\partial_a+(j-1)c\partial_b+(j-2)d\partial_c+\ldots\}I-(ij-2w)I.$$

If in this formula we write

$$O=jb\partial_a+(j-1)c\partial_b+(j-2)d\partial_c+\ldots,$$

it becomes $\qquad (jP-Q)I=aOI-(ij-2w)bI,$

which, when I is a satisfied invariant, so that $ij-2w=0$ and $OI=0$, reduces to $\qquad (jP-Q)I=0,$

showing that the forms obtained by operating with either P or Q on any satisfied invariant are the same to a numerical factor près.

Now, each quadratic protomorph is a satisfied invariant (for when $w=j$ and $i=2$, $ij-2w=0$), and therefore the cubic protomorphs found by operating on the quadratic ones with Q will only differ by a numerical factor from those already obtained by the operation of P. But we must not conclude from this that the cubic protomorphs are unique. Their number is in fact given by the formula $\qquad (j;3,j)-(j-1;3,j),$

where it is obvious that

$$(j-1;3,j)=(j-1;3,j-1);$$

so that the above formula may be written

$$(j;3,j)-(j-1;3,j-1),\ \text{or say}\ \Delta(j;3,j).$$

Now, there is a simple rule for finding $(j;\ 3,j)$; it is the nearest integer to $\frac{(j+3)^2}{12}$. From the following table, obtained by the use of this rule,

$j=$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$(j;3,j)=$	2	3	4	5	7	8	10	12	14	16	19	21	24	27
$\Delta(j;3,j)=$		1		1		1		2		2		2		3

it may be seen that for any odd number $j = > 9$ there are two or more forms of extent j equally entitled to rank as protomorphs. If l be the last letter which

occurs in one of these forms, its first term will of course be a^2l; the difference between any two such forms will not involve the letter l, and will only extend to k, but will still be of the same (potential) extent as l.

The property of the protomorphs $a, P_2, P_3, P_4, \ldots$ is that every invariant is a rational integral function of them divided by some power of a, as appears from the fact that Q, any given rational integral function whatever of the letters $a, b, c, d, e, \ldots$, may obviously be expressed as a rational integral function of $a, b, P_2, P_3, P_4, \ldots$ divided by some power of a. Thus,

$$Q = a^{-m}\phi(a, b, P_2, P_3, P_4, \ldots).$$

Suppose Q to be an invariant I; then

$$I^m = \phi(a, b, P_2, P_3, P_4, \ldots),$$

and, consequently,

$$\Omega(Ia^m) = \frac{d\phi}{da}\Omega a + \frac{d\phi}{db}\Omega b + \frac{d\phi}{dP_2}\Omega P_2 + \frac{d\phi}{dP_3}\Omega P_3 + \ldots,$$

where Ω is the annihilator for invariants; so that

$$\Omega(Ia^m) = 0, \ \Omega a = 0, \ \Omega P_2 = 0, \ \Omega P_3 = 0, \ldots$$

We have therefore
$$\frac{d\phi}{db}\Omega b = a\frac{d\phi}{db} = 0.$$

Hence ϕ does not contain b, but is a rational integral function of the protomorphs alone, and

$$I = a^{-m}\phi(a, P_2, P_3, P_4, \ldots).$$

I shall show how to obtain a similar scale of forms possessing like properties for pure reciprocants.

LECTURE XV.

A Protomorph may be defined as a form whose weight is equal to its actual extent, so that its type is $j; i, j$. The first protomorph is a, which corresponds to $j = 0$. For higher values of j it follows immediately from the definition that every protomorph will contain a term $a^{i-1}l$, in which the letter of highest extent appears only in the first degree multiplied by a power of the first letter. The existence of this term enables us to instantly recognize a protomorph. As in the case of invariants, it will be shown that every pure

reciprocant is either a rational integral function of protomorphs or else such a function divided by some power of a. But first it will be better to prove *à priori* their existence and exhibit examples of them for the earlier values of j.

It was proved, in Lecture IX, that the number of pure reciprocants of the type $w; i, j$ is at least equal to

$$(w; i, j) - (w - 1; i + 1, j).$$

Now, obviously, the number of partitions of w into i parts not exceeding $w + \varepsilon$ is the same as the number of partitions of w into i parts not exceeding w, so that

$$(w; i, w + \varepsilon) = (w; i, w);$$

and since, by a well-known theorem, $(w; i, j) = (w; j, i)$, we see that

$$(w; w + \varepsilon, j) = (w; j, w + \varepsilon) = (w; j, w) = (w; w, j),$$

a result which follows more immediately from the consideration that the partitions of $w; w + \varepsilon, j$ differ only from those of $w; w, j$ by ε columns of zeros, as we see in the annexed example:

3; 5, 3	3; 3, 3
30000	300
21000	210
11100	111

Hence, if $w = j$, and $i => j$, we have

$$(w; i, j) = (j; j, j)$$

and

$$(w - 1; i + 1, j) = (j - 1; j - 1, j - 1).$$

Thus, the number of pure reciprocants of the type $j; j, j$ is

$$(j; j, j) - (j - 1; j - 1, j - 1),$$

in other words, the difference between the indefinite partitions of j and those of $j - 1$. Expressed by means of generating functions, this difference is the coefficient of x^j in

$$\frac{1 - x}{(1 - x)(1 - x^2)(1 - x^3) \ldots. (1 - x^j)}$$

= coefficient of x^j in the expansion of

$$\frac{1}{(1 - x^2)(1 - x^3) \ldots. (1 - x^j)}.$$

This coefficient is a positive integer for all values of j (except $j = 1$, when it is zero), which proves the existence of reciprocants of the type $j; j, j$ when j has any value except unity.

But we wish to prove the existence of one or more reciprocants of the type $j; j.j$ which actually contain a term of the form $a^{j-1}l$, where the letter l is of extent j. The number of such forms is the difference between the number of pure reciprocants of the types $j; j, j$ and $j; j, j-1$.

Now, the number of linearly independent pure reciprocants of the type $j; j, j$ has just been shown to be

$$(j; j, j) - (j-1; j-1, j-1).$$

And, in like manner, that of the linearly independent reciprocants of the type $j; j, j-1$ is

$$\begin{aligned}&(j; j, j-1) - (j-1; j+1, j-1)\\ = &(j; j, j-1) - (j-1; j-1, j-1).\end{aligned}$$

The difference between these two numbers is therefore

$$(j; j, j) - (j; j, j-1) = 1.$$

For the only partition not common to the two types is $j.0^{j-1}$, made up of one j and $j-1$ zeros, which belongs to the first type, but not to the second. Hence reciprocants of the type $j; j, j$ contain one term which those of the type $j; j, j-1$ do not, and which can only be $a^{j-1}l$. This proves the existence of protomorphs.

In the latter part of the above proof we have assumed the truth of the theorem, which, however probable, is not demonstrated, that the number of reciprocants of the type $w; i, j$ is $(w; i, j) - (w-1; i+1, j)$ and *no more* [that concerns the subtrahend, viz.: $(j: j, j-1) - (j-1; j-1, j-1)$].

We shall, however, have an independent method of arriving at Protomorphs by direct generation, just as we saw that all the cubic protomorphs to invariants were derivable by direct operation of generators from the quadratic ones.

The difference between the two cases is that the lowest degree of Invariantive Protomorphs fluctuates alternately between 2 and 3. For Reciprocantive Protomorphs the lowest degree corresponding to a given extent fluctuates, but has a tendency to rise, and goes on progressing until it exceeds any assignable number.

It is interesting to find what the degrees are for successive values of j. The calculations required are greatly facilitated by an extensive table of partitions given by Euler in 1750, and partly reproduced by Cayley in the *American Journal of Mathematics*, Vol. IV, Part 3. In the table as presented by Cayley, the number in column j and line i means the number of ways of partitioning j into exactly i parts (zeros excluded). Hence, to find the number of ways of

partitioning j into i parts or fewer; *i. e.* to find $(j;\, i,\, \infty)$ or its equivalent $(j;\, i,\, j)$, we must add up the numbers in the 1st, 2d, 3d, ... ith lines of column j.

When these summations are made we obtain the subjoined table:

EXTENT $j =$

DEGREE $i =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10
3	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37
4	1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84
5	1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84	101	119	141
6	1	1	2	3	5	7	11	14	20	26	35	44	58	71	90	110	136	163	199
7	1	1	2	3	5	7	11	15	21	28	38	49	65	82	105	131	164	201	248
8	1	1	2	3	5	7	11	15	22	29	40	52	70	89	116	146	186	230	288

The number of pure reciprocants of the type $j;\, i,\, j$ is

$$(j;\; i, j) - (j-1;\; i+1, j) = (j;\; i, j) - (j-1;\; i+1, j-1).$$

To find the minimum degree for protomorphs of extent j we have therefore only to see for what value of i any figure in the j column first becomes greater than the figure in the column to the left one place lower down. The fluctuations of the minimum degree are indicated by the dark irregularly waving line which runs through the table.

Accordingly, we find that the types of the protomorphs, omitting w, which is always equal to j, are as follows:

(2, 2), (3, 3), (3, 4), (4, 5), (3, 6), (4, 7), (4, 8), (5, 9), (5, 10), (5, 11), (5, 12), ...,

whereas for invariants they are

(2, 2), (3, 3), (2, 4), (3, 5), (2, 6), (3, 7), (2, 8), (3, 9), (2, 10), (3, 11), (2, 12), ...

Corresponding to the extents

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...,

the lowest degrees of the Reciprocantive Protomorphs are

2, 3, 3, 4, 3, 4, 4, 5, 5, 5, 5, ...

Contrast this with the regularly fluctuating series

2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, ...,

which shows the minimum degrees of invariantive protomorphs for successive extents.

It may be proved, from known formulae in the theory of partitions, that as the extent increases the minimum degree of reciprocantive protomorphs increases (on the whole) and ultimately becomes infinite when the extent is so.

The apparent number of protomorphs to the several types is

(2, 2),	(3, 3),	(3, 4),	(4, 5),	(3, 6),	(4, 7),	(4, 8),	(5, 9),	(5, 10),	(5, 11),	(5, 12),	....
1	1	1	1	1	1	2	3	4	2	3	

The explanation of this multiplicity is the same as that previously given for the case of invariants: the difference between any two protomorphs of a given type $j;\ i, j$ will be a reciprocant (no longer a protomorph) of the type $j;\ i, j-1$.

For the only term containing the letter l (of extent j) will disappear from the result of subtraction; and, accordingly, the above numbers, each diminished by unity, will give the numbers of a set of reciprocants of the same degree-weight as the protomorphs, but of a smaller (actual) extent.

Assuming that the number of pure reciprocants of the type $w;\ i, j$ is correctly given by the formula

$$(w;\ i, j) - (w-1;\ i+1, j),$$

Euler's great table of partitions, already referred to, enables us to carry on the determination of the minimum degree and multiplicity of protomorphs for all extents as far as 59.

If m is the multiplicity corresponding to the minimum degree i of a reciprocantive protomorph whose extent is j, we form without difficulty, using only the principles explained above, the following table:

$j=$	0	1	2	3	4	5	6	7	8	9	10	11
$i=$	1	–	2	3	3	4	3	4	4	5	5	5
$m=$	1	0	1	1	1	1	1	1	2	3	4	2 ,

$j=$	12	13	14	15	16	17	18	19	20	21	22	23
$i=$	5	6	6	6	6	7	7	7	7	7	8	8
$m=$	3	6	8	5	5	15	18	12	12	2	40	32 ,

$j=$	24	25	26	27	28	29	30	31	32	33	34	35
$i=$	8	8	8	9	9	9	9	10	10	10	10	10
$m=$	32	14	6	84	82	58	45	207	211	180	161	102 ,

$j=$	36	37	38	39	40	41	42	43	44	45	46	47
$i=$	10	11	11	11	11	11	11	12	12	12	12	12
$m=$	45	482	469	391	320	167	13	1126	1064	881	687	337

,

$j=$	48	49	50	51	52	53	54	55	56	57	58	59
$i=$	13	13	13	13	13	13	13	14	14	14	14	14
$m=$	2829	2666	2492	2097	1643	892	26	6394	6017	5227	4266	2755

.

Notice the repetitions of i indicated by the series

$$1^1,\ 0^1,\ 2^1,\ 3^2,\ 4^1,\ 3^1,\ 4^2,\ 5^4,\ 6^4,\ 7^5,\ 8^5,\ 9^4,\ 10^6,\ 11^6,\ 12^5,\ 13^7,\ 14^{5+?}.$$

It will be observed that there is a general tendency of the number of equal values of i to increase, but that this is subject to occasional fluctuations. When $j=5$, $i=4$; but when $j=6$, $i=3$, so that the minimum value of i recedes. After this point is reached, i either advances or remains stationary, but never recedes.

In order actually to find the protomorphs, we may use the annihilator V. This was my original method of obtaining them; a shorter way, analogous to that used by Halphen for differential invariants (principiants), has been previously mentioned, but it will be instructive to begin with the method of indeterminate coefficients. In the first place we have the form a of weight 0, which is annihilated by

$$V = 2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\,\partial_d + (7ad + 7bc)\,\partial_e + \ldots.$$

For weight 1 there is no pure reciprocant. We could not make $R = \lambda a^{i-1}b$, for then $VR = 2\lambda a^{i+1}$, which cannot vanish unless $\lambda = 0$ and consequently $R = 0$.

To find the Protomorph of extent 2, assume $R = \lambda ac + \mu b^2$; then

$$VR = 4\mu a^2b + 5\lambda a^2b = (4\mu + 5\lambda)\,a^2b.$$

Hence λ and μ are proportional to 4 and -5, and we may write

$$R = 4ac - 5b^2.$$

For extent 3, assuming $R = \lambda a^2d + \mu abc + \nu b^3$, we have

$$VR = 2\mu a^3c + 6\nu a^2b^2 + 5\mu a^2b^2 + 6\lambda a^3c + 3\lambda a^2b^2,$$

which vanishes when

$$2\mu + 6\lambda = 0,\quad 6\nu + 5\mu + 3\lambda = 0.$$

We may therefore write $\lambda = 1$, $\mu = -3$, $\nu = 2$, and thus obtain

$$R = a^2d - 3abc + 2b^3.$$

For extent 4 the table of minimum degrees indicates the existence of a protomorph of degree 3. To find its value we assume

$$R = xa^2e + \lambda abd + \mu ac^2 + \nu b^2c.$$

Operating with V, we find

$$\begin{array}{rccc} & a^2d & a^2bc & ab^3 \\ VR = & 2\lambda & 4\nu & . \\ & . & 10\mu & 5\nu \\ & . & 6\lambda & 3\lambda \\ & 7x & 7x & . \end{array}$$

In order that VR may vanish, we must have

$$2\lambda + 7x = 0,\ 4\nu + 10\mu + 6\lambda + 7x = 0,\ \text{and}\ 5\nu + 3\lambda = 0.$$

To avoid fractions, let $x = 50$; then $\lambda = -175$, $\nu = 105$, and $\mu = 28$; thus,

$$R = 50a^2e - 175abd + 28ac^2 + 105b^2c;$$

whereas, the protomorph of extent 4 for Invariants is $ae - 4bd + 3c^2$. There is no reciprocant of degree 2 weight 4 to correspond to this.

LECTURE XVI.

By using the generator for pure reciprocants instead of the annihilator V, we readily obtain the protomorph of extent 5 and of the fourth degree whose existence is indicated in the previously given table of minimum degrees. We have only to operate on the protomorph of degree 3 and extent 4 with

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + 7(af - be)\partial_e + \ldots.$$

Thus,

$$\begin{aligned} &G(50a^2e - 175abd + 28ac^2 + 105b^2c) \\ &= 4(ac - b^2)(-175ad + 210bc) \\ &+ 5(ad - bc)(56ac + 105b^2) \\ &+ 6(ae - bd)(-175ab) \\ &+ 7(af - be)(50a^2). \end{aligned}$$

Rejecting the numerical factor 35, which is common to all the terms in the result, and at the same time writing the terms themselves in reverse order, we have

$$\begin{aligned} &10a^2(af - be) - 30ab(ae - bd) + (ad - bc)(8ac + 15b^2) + 4(ac - b^2)(-5ad + 6bc) \\ &= 10a^3f - 40a^2be - 12a^2cd + 65ab^2d + 16abc^2 - 39b^3c, \end{aligned}$$

which is the protomorph in question.

The form just found is irreducible, as indeed it ought to be, since the minimum degree for extent 5 is greater than that for extent 4 by unity, which exactly corresponds with the unit increase of degree due to the operation of G. But if we use G to generate a protomorph of extent 4 from that of extent 3, the resulting form will be reducible. In fact,

$$\begin{aligned}
&G(a^2d - 3abc + 2b^3) \\
&= 4(ac - b^2)(-3ac + 6b^2) + 5(ad - bc)(-3ab) + 6(ae - bd)a^2 \\
&= 3(2a^3e - 7a^2bd - 4a^2c^2 + 17ab^2c - 8b^4).
\end{aligned}$$

If now we write

$$\begin{aligned}
&ac - \tfrac{5}{4}b^2 = M, \\
&a^2d - 3abc + 2b^3 = A, \\
&a^3e - \tfrac{7}{2}a^2bd - 2a^2c^2 + \tfrac{17}{2}ab^2c - 4b^4 = B,
\end{aligned}$$

we have shown that

$$GA = 6B.$$

But

$$\begin{aligned}
50B + 128M^2 &= 25(2a^3e - 7a^2bd - 4a^2c^2 + 17ab^2c - 8b^4) + 8(4ac - 5b^2)^2 \\
&= a(50a^2e - 175abd + 28ac^2 + 105b^2c);
\end{aligned}$$

so that B is reducible, being expressible as a rational integral function of a, M, and the previously obtained protomorph of degree 3 and extent 4.

The general theory of the generator G is contained in that of the differentiation of absolute reciprocants, in which, if $\mu = 3i + w$, where w is the weight and i the degree of any pure reciprocant R, we have

$$\frac{R}{a^{\frac{\mu}{3}}} = \pm \frac{R_1}{a_1^{\frac{\mu}{3}}},$$

and, consequently,

$$\frac{d}{dx}\left(\frac{R}{a^{\frac{\mu}{3}}}\right) = \pm \frac{dy}{dx}\cdot\frac{d}{dy}\left(\frac{R_1}{a_1^{\frac{\mu}{3}}}\right),$$

where R_1 and a_1 are what R and a become when x and y are interchanged. Hence

$$\frac{a\dfrac{dR}{dx} - \dfrac{\mu}{3}R\dfrac{da}{dx}}{a^{\frac{\mu}{3}+1}},$$

and therefore also the numerator of this fraction is a reciprocant.

Remembering that

$$\frac{da}{dx} = 3b, \quad \frac{db}{dx} = 4c, \quad \frac{dc}{dx} = 5d, \ldots,$$

the numerator may be written

$$a\frac{dR}{dx} - \mu bR = GR.$$

The ordinary expression for G is found by writing

$$\begin{aligned} a\frac{d}{dx} - \mu b = {} & a(3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots) \\ & - b(3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots). \end{aligned}$$

If the actual extent of R is j, that of GR is $j+1$; for the operation of G introduces an additional letter. Both the weight and degree are also increased by unity. Thus, the type of R being w; i, j, that of GR is $w+1$; $i+1, j+1$. Suppose the weight of R to be equal to its actual extent; then R is a protomorph of the type j; i, j, and GR, whose type is $j+1$; $i+1$, $j+1$, is also a protomorph. This proves the existence of protomorphs for every possible extent. Starting with the form $4ac - 5b^2$ we obtain, by successive eduction, a series of protomorphs of the type j; j, j for which the general expression is

$$G^{j-2}(4ac - 5b^2),$$

where j has any of the values $2, 3, 4, \ldots$.

If R is a protomorph of minimum degree, GR (if irreducible) will also be a protomorph of minimum degree. Hence the minimum degree can never increase by more than one unit when the extent is increased by unity.

The second educt G^2R is always reducible; for

$$\begin{aligned} G^2R &= \left\{a\frac{d}{dx} - (\mu+4)b\right\}\left(a\frac{d}{dx} - \mu b\right)R \\ &= \left\{a^2\frac{d^2}{dx^2} - (2\mu+1)ab\frac{d}{dx} - 4\mu ac + \mu(\mu+4)b^2\right\}R. \end{aligned}$$

Combining this with $M = ac - \frac{5}{4}b^2$, we have

$$5G^2R + 4\mu(\mu+4)MR = a\left\{5a\frac{d^2}{dx^2} - 5(2\mu+1)b\frac{d}{dx} + 4\mu(\mu-1)c\right\}R,$$

where the right-hand side is divisible by a, showing that the degree of G^2R is always depressible by unity. R being a protomorph of degree i and extent j,

$$\left\{5a\frac{d^2}{dx^2} - 5(2\mu+1)\frac{d}{dx} + 4\mu(\mu-1)c\right\}R$$

is one of degree $i+1$ and extent $j+2$. Hence we may conclude that an increase in the minimum degree for protomorphs cannot be immediately followed by another increase; for, if this were possible, the minimum degree for extent $j+2$ would be $i+2$, instead of being $i+1$ at most.

This conclusion is in accordance with the sequence of the values of i in the table of minimum degrees, and as far as it goes confirms the exactitude of the formula $(w; i, j) - (w-1; i+1, j)$ for the number of pure reciprocants which was assumed in calculating the table.

The method previously employed to prove that every invariant is a rational integral function of protomorphs, or such function divided by a power of a, may be very easily extended to the case of reciprocants.

In the first place, it is obvious that every rational integral function of the letters $a, b, c, d, \ldots$ is by successive substitutions reducible to the form

$$a^{-\theta}\Phi(a, b, P_2, P_3, P_4, \ldots P_j),$$

where P_j means the protomorph of extent j.

Let any reciprocant R be put under this form; then

$$a^{\theta}R = \Phi(a, b, P_2, P_3, P_4, \ldots P_j),$$

and, consequently,

$$V(a^{\theta}R) = \frac{d\Phi}{da}Va + \frac{d\Phi}{db}Vb + \frac{d\Phi}{dP_2}VP_2 + \ldots + \frac{d\Phi}{dP_j}VP_j.$$

Now, V annihilates $R, a, P_2, P_3, \ldots P_j$, since these are all pure reciprocants. Hence the above identity reduces to $\frac{d\Phi}{db}Vb = 0$, from which (since Vb does not vanish) we conclude that Φ does not contain b explicitly. Thus,

$$a^{\theta}R = \Phi(a, P_2, P_3, P_4, \ldots P_j),$$

and the theorem is established for reciprocants.

The Protomorphs for Reciprocants as far as extent 8 are as follows:

$$P_2 = 4ac - 5b^2,$$

$$P_3 = a^2d - 3abc + 2b^3,$$

$$P_4 = 50a^2e - 175abd + 28ac^2 + 105b^2c,$$

$$P_5 = 10a^3f - 40a^2be - 12a^2cd + 65ab^2d + 16abc^2 - 39b^3c,$$

$$P_6 = 14a^2g - 63abf - 1350ace + 1782b^2e + 1470ad^2 - 4158bcd + 2310c^3,$$

$$P_7 = 7a^3h - 35a^2bg - 539a^2cf + 735ab^2f + 605a^2de + 306abce - 1485b^3e \\ - 2135abd^2 + 1001ac^2d + 3465b^2cd - 1925bc^3,$$

$$P_8 = 420a^3i - 2310a^2bh - 25648a^2cg + 9240a^2df + 21780a^2e^2 + 36680ab^2g \\ + 85386abcf - 191730abde - 59220ac^2e + 120540acd^2 \\ - 126945b^3f + 252126b^2ce + 169260b^2d^2 - 419034bc^2d \\ + 129360c^4.$$

The work necessary for obtaining the first four of these, P_2, P_3, P_4, P_5, has been fully set out. Since P_4 is of degree 3, its second educt, G^2P_4, is of degree 5 and its reduced second educt of degree 4. A linear combination of this with a form whose leading term is a^2ce becomes divisible by a and gives P_6; but as this requires the preliminary calculation of the form (a^2ce), it is simpler to find P_6 directly by the method of indeterminate coefficients, and thence by eduction to get P_7 and P_8. Thus (to a numerical factor près) P_7 is the educt and P_8 the reduced second educt of P_6. Beyond this point the calculation of protomorphs has not at present been carried.

Referring to the table which gives the minimum degree and multiplicity for a Protomorph of any extent, we see that the multiplicity exceeds unity when the extent $j => 8$, and is exactly equal to 2 when $j = 8, 11,$ or 21.

Hence the protomorphs as far as P_7 inclusive are unique; but there are two forms of extent 8 and degree 4 any linear combination of which (provided it contains the term a^3i) may be regarded as a protomorph. One of these forms is P_8, whose value is given above; the other is a linear combination of P_8 with a form, whose leading term is a^2cg, hereafter to be set forth.

The irreducible forms for extent 2 are a and P_2; every other form must be simply a power of P_2 multiplied by a power of a. We proceed to the calculation of all the Irreducible Forms for the extents 3 and 4 respectively. When $j = 3$, we may combine the protomorphs

$$4ac - 5b^2$$

and

$$a^2d - 3abc + 2b^3$$

with one another.

Adding 125 times the square of the latter to 4 times the cube of the former and dividing by a, there results the form

$$125a^3d^2 - 750a^2bcd + 500b^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c.$$

This form is analogous to the discriminant of the cubic, but is of a higher degree by one unit. Its type is 6; 5, 3, whereas that of the discriminant is 6; 4, 3.

In the case of invariants, we have to combine $ac - b^2$ with $a^2d - 3abc + 2b^3$. The square of the second, added to 4 times the cube of the first, gives

$$a^4d^2 - 6a^3bcd + 4a^2b^3d + 9a^2b^2c^2 - 12ab^4c + 4b^6 + 4a^3c^3 - 12a^2b^2c^2 + 12ab^4c - 4b^6.$$

Here the term $12ab^4c$ is nullified by $-12ab^4c$, so that the result contains a^2, the other factor being the discriminant

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2,$$

which is of the type 6; 4, 3.

We may show *à priori*, assuming the problematical but highly probable formula $(w;\ i, j) - (w - 1;\ i + 1, j)$, that the type 6; 4, 3 does not belong to any reciprocant.

For, as seen in the partitionments set out below,

$$(6;\ 4,\ 3) - (5;\ 5,\ 3) = 5 - 5 = 0$$

3.3	3.2
3.2.1	3.1.1
3.1.1.1	2.2.1
2.2.2	2.1.1.1
2.2.1.1	1.1.1.1.1

We can by no other means combine the protomorphs with one another or with the Quasi-Discriminant ($125a^3d^2 \ldots .$) so as to obtain additional fundamental forms. Every Rational Integral Pure Reciprocant of extent 3 is therefore necessarily a rational integral function of the four forms

deg. wt.

1.0 a,

2.2 $4M = 4ac - 5b^2$,

3.3 $A = a^2d - 3abc + 2b^3$,

5.6 $(a^3d^2) = 125a^3d^2 - 750a^2bcd + 500b^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c$.

These are connected by a syzygy of degree-weight 6.6, viz.:

$$125A^2 + 256M^3 = a\,(a^3d^2),$$

analogous to the syzygy of the same degree-weight, in the Theory of the Binary Cubic, which connects the Discriminant with a and the Protomorphs of extent 2 and 3.

It will be clearly seen from an inspection of the fundamental forms that there is no law for the coefficients of Reciprocants akin to that of their algebraical sum being zero in Invariants.

(*To be continued.*)

A New Method in Analytic Geometry.

By William E. Story.

About a year ago, in connection with a course of lectures on plane cubic curves, I had occasion to prove that the four tangents to a non-singular plane cubic from any point of the same have a constant anharmonic ratio, and an instantaneous proof then occurred to me which is certainly, having a purely algebraic basis, not open to the objection which Sturm* has raised (without any real foundation, as it seems to me) to that given by Salmon. Shortly afterwards I applied the same method to the proof of other geometrical theorems, and gave it a general form in my own mind, at least, but have only recently found leisure to put it into shape for publication. It may not be uninteresting as an application to geometry of the fundamental theorem of algebra briefly stated: every equation has a root.

Let $f(\alpha, x) = 0$ be an equation concerning whose left member $f(\alpha, x)$ we make the following assumptions:

a). It *certainly* involves a variable α, being a rational algebraic polynomial in α.

b). It *possibly* involves a second variable x, and if so, is a rational algebraic polynomial in x.

c). It has no factor involving x but not α.

Under these assumptions $f(\alpha, x)$ may break up into factors, some of which involve α alone, and some both α and x, but this is of no consequence.

There are then two cases:

I. If the equation involve both α and x, for every assumed value of x it defines a certain finite number (its degree in α) of values of α, which may be

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* Crelle's Journal, Vol. 90, p. —.

considered so many functions of x, some of which will necessarily vary with the assumed value of x, and for every assumed value of α it determines a certain finite number (its degree in x) of values of x.

II. If the equation involve α but not x, it defines a certain finite number (its degree in α) of values of α, which are the same, whatever value x may be assumed to have, *i. e.* which are constant.

Here we make no distinction between real and imaginary values. If, then, it is known that the equation will not be satisfied by any real or imaginary value of x when a certain value is assigned to α, it must come under the second case, and the values of α determined by it are constant.

The following extension of this theorem is evident. If a quantity α is connected with k other quantities $x, y, z, \ldots.$ by an equation $f(\alpha, x, y, z, \ldots.) = 0$, say $f = 0$, whose left member is a rational algebraic polynomial in these quantities (certainly involving α and possibly $x, y, z, \ldots.$), and if $x, y, z, \ldots.$ are connected among themselves by $k - 1$ auxiliary equations $\phi = 0$, $\psi = 0, \ldots.$; then, if the equations $f = 0$, $\phi = 0$, $\psi = 0, \ldots.$ cannot be satisfied by any set of real or imaginary values of $x, y, z, \ldots.$ when a certain value is assigned to α, the values of α determined by $f = 0$ will be constant, whatever values may be assigned to $x, y, z, \ldots.$ satisfying the auxiliary equations. For, by means of the auxiliary equations, $k - 1$ of the quantities $x, y, z, \ldots.$ may be eliminated from the equation $f = 0$, which is thus reduced to an equation between α and one of the other quantities, say x, to which the previous theorem is applicable.

This extended theorem can be applied to the proof of numerous geometrical theorems. Let $x, y, z, \ldots.$ be the co-ordinates of a variable element (point, straight line or plane) of a one-way algebraic locus, *i. e.* point of a plane or twisted curve, tangent of a plane or twisted curve, generator of a ruled surface, edge of a cone, tangent plane of a cone or developable surface; let $\phi = 0$, $\psi = 0, \ldots.$ be the equations of the locus, together with whatever identities exist between the co-ordinates (*e. g.* such an identity exists between the six co-ordinates of a straight line); and let there be a geometrical magnitude α determined by any position of the variable element and, it may be, certain fixed elements or geometrical forms (curves and surfaces), such that α is capable of determination by an algebraic equation which, when rationalized and cleared of fractions, is $f(\alpha, x, y, z, \ldots.) = 0$, say of the degree ν in α. This rationalized equation defines ν geometrical magnitudes determined by each element of the

locus, or say α stands in any one of ν geometrical relations to a given element $x, y, z, \ldots$. If, then, it is found that a certain value stands in neither of these relations to any real or imaginary element of the locus, the above theorems show that the several geometrical magnitudes determined by an element of the locus are constant; *i. e.* that, taken in some order or other, they have the same values for all elements of the locus. This will be more evident from the examples which I give below. To prove a given theorem involving the constancy of a magnitude α, all that has to be done is to find a value of α which can be shown to be impossible for any real or imaginary element of the locus, and the readiest manner of doing this seems to be to assume some simple value, such as 0 or ∞, and prove its impossibility. In a large number of cases the very nature of the locus will make this certain, and in these cases this method is instantaneous, furnishing, it seems to me, the simplest conceivable proof. In other cases the assumed value of α, say 0 or ∞, will be possible only when the expression of α assumes an indeterminate form, and then it will be necessary to show that the true value is not the assumed value. The first three applications given below belong to the first class, the others to the second class of cases. In both these classes of cases α can be expressed (not necessarily rationally) in terms of the co-ordinates of the element of the locus. It would be interesting to find a case in which α could not be so expressed, but to which the method is still applicable.

The auxiliary equations may evidently constitute a restricted system, as in the case of the twisted cubic curve, without affecting the applicability of the method.

From the readiness with which one application suggests another, it seems probable that the method may be useful in the discovery of new theorems.

It may be useful here to give an example to which the method is not applicable, for the purpose of showing the cause of failure. That the sum of the focal distances of a point of an ellipse is constant cannot be so proved, for the *rationalized* equation which determines the sum of these distances determines also their difference, and if the sum is constant, the difference will vary, and *vice versa*, and it will not be possible to show that neither sum nor difference can have a certain value, say 0; in fact, the ordinary method of proof of the theorem stated holds only for *real* points of the ellipse, while the curve defined as the locus of points the sum of whose distances from two given points is a given constant will be an ellipse only if that constant is greater than the distance

between the given points. In a certain sense we may say that, for a part of the curve, including the real part, the sum, and for the rest the difference, of the focal distances is constant.

One of the simplest cases is that in which α is the anharmonic ratio of four points, lines or planes determined by the variable element of the locus. There are six values of this anharmonic ratio, depending upon the order in which the four points, lines or planes are taken, so that the order of the rationalized equation $f = 0$ is 6; and to prove the constancy of these values for all elements of the locus by our method, it would be necessary to show that there is a value which *neither* of these anharmonic ratios can have for any real or imaginary element of the locus.

As a matter of form I call attention here, once for all, to the evident fact, already mentioned incidentally, that in all the following applications the quantity α, whose constancy is to be proved, is connected with the co-ordinates of the variable element of the given locus by a rational algebraic equation, as is necessary for the proof.

1. The anharmonic ratio α of the junctions of a variable point of a conic to four fixed points of the same, taken in a given order, is constant. For it can have the value 0 only if two of the junctions coincide, *i. e.* only if the variable point lies on the junction of two of the fixed points; but, the curve being of the second order, the two fixed points are the only points of the conic on their junction; therefore $\alpha = 0$ only if the variable point coincides with one of the fixed points; but then one of the four junctions in question is the tangent at this fixed point and the other three junctions are lines differing from each other and from this tangent; so that, even in this case, α is not 0, and therefore is constant for every position of the variable point.

2. The four tangents to a non-singular plane cubic curve from any point of the same (other than the two coincident tangents at the point) have a constant anharmonic ratio α (*i. e.* there are six constant values of α, according to the order in which the tangents are taken). For $\alpha = 0$ only when two of these tangents coincide, which they never do; for the curve, being of the third order, has no double tangent, and an inflexional tangent meets the curve only at its point of contact and counts for only one of the four tangents from this point.

3. The four planes joining a variable tangent of a twisted cubic to four fixed points of the same, taken in a given order, have a constant anharmonic ratio α.

For $\alpha = 0$ only if two of these planes coincide, *i. e.* only if the tangent lies in the same plane with two of the fixed points; this plane will then meet the curve in the two fixed points and in two points at the contact of the tangent, *i. e.* in four points in all, which is impossible, the curve being of the third order; or the point of contact of the tangent will be one of the fixed points, in which case one of the planes is the osculating plane at this point and does not coincide with either of the other three planes.

The reciprocals of these three theorems are known to be true by the principle of duality. These reciprocals are:

The anharmonic ratio α of the intersections of a variable tangent of a conic with four fixed tangents of the same, taken in a given order, is constant.

The four intersections of a non-singular plane curve of the third class (which is of the sixth order) with any tangent of the same (other than two coincident intersections at the point of contact of the tangent) have a constant anharmonic ratio α (*i. e.* there are six constant values of α, according to the order in which the points are taken).

The four points of intersection of a variable generator of a developable surface of the third class with four fixed tangent planes of the same, taken in a given order, have a constant anharmonic ratio α.

4. The angle between the junctions of a variable point of a circle with two fixed points of the same, taken in a given order, is constant. Let $P(x, y)$ be the variable point, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ the fixed points, ϑ the angle which PP_2 makes with PP_1, r the radius of the circle, and the centre of the circle in the origin of co-ordinates; then $x^2 + y^2 = r^2$,

$$\tan\vartheta = \frac{(y_1 - y_2)x + (x_2 - x_1)y + x_1 y_2 - x_2 y_1}{x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1 x_2 + y_1 y_2},$$

which can be 0 only when P lies on the junction of P_1P_2, or when P is infinite (*i. e.* one of the circular points). If P lies on P_1P_2 it must coincide with P_1 or P_2, since these are the only points of the circle on this line; but if it coincides with P_1 the junctions in question are the tangent at P_1 and the line P_1P_2, and the tangent of the angle between these lines is not 0. If P is a circular point, $y = \pm ix$, $x = \infty$,

$$\tan\vartheta = \frac{y_2 - y_1 \mp i(x_2 - x_1)}{x_1 + x_2 \pm i(y_1 + y_2)},$$

provided P_1 and P_2 are finite, and this value of $\tan\vartheta$ is evidently not 0. If P or P_2 is a circular point, evidently $\tan\vartheta = i$.

5. The segment cut out of a variable tangent to a conic by two fixed tangents of the same subtends a constant angle at either focus. This is, as is well known, readily proved by the reciprocal of number 1 above, but it may be proved independently as follows: Let P be the focus, P_1 and P_2 the intersections of the variable tangent with the two fixed tangents, and ϑ the subtended angle; then, from the general expression given in the last number, it is evident that $\tan\vartheta$ will be 0, since P is finite, only if P, P_1 and P_2 lie in one straight line, *i. e.* only if the variable tangent passes through the intersection of the fixed tangents, and therefore coincides with one of them, in which case the angle ϑ is the angle between the lines joining P to the intersection of the fixed tangents and to the point of contact of one of them, whose tangent is evidently not 0. If one of the fixed tangents passes through the focus, $\tan\vartheta$ is evidently constantly $= i$.

6. The ratio of the distances of a variable point of a conic from either focus and the corresponding directrix is constant. We define the focus as the intersection of two circular tangents, *i. e.* tangents each of which passes through a circular point, and the corresponding directrix as its polar, *i. e.* the chord of contact of tangents from it. Evidently this ratio will be 0 only when the distance from the focus is 0 or that from the directrix is infinite, *i. e.* only when the variable point is the point of contact of one of the tangents from the focus, or when the variable point is infinite. If the variable point is infinite, the ratio in question is evidently the cosecant of the angle between the directrix and the direction of the infinite point, *i. e.* is not in general 0. If the variable point is the point of contact of one of the tangents from the focus, then its focal distance is 0, as is also its distance from the directrix on which it lies, and the ratio in question is undetermined. For a point very near the point of contact, both distances are small of the same order and their ratio is determinate. Let the origin be the focus and axis of x perpendicular to the directrix, whose equation is then of the form $x + d = 0$; the point $(-d, id)$ is then the point of contact of one of the tangents from the focus, and the ratio in question is $\frac{\sqrt{x^2+y^2}}{x+d}$ for this point. For a neighboring point $x = -d + \delta$, $y = id + \left(\frac{\partial y}{\partial x}\right)\delta + \frac{1}{2}\left(\frac{\partial^2 y}{\partial x^2}\right)\delta^2$, where $\left(\frac{\partial y}{\partial x}\right)$ and $\left(\frac{\partial^2 y}{\partial x^2}\right)$ are to be taken for the point of contact. Evidently $\left(\frac{\partial y}{\partial x}\right) = -i$, and $\left(\frac{\partial^2 y}{\partial x^2}\right)^2$ has a value different from 0, if the point of contact is finite. For this

neighboring point then $y = i(d - \delta) + \frac{1}{2}\left(\frac{\partial^2 y}{\partial x^2}\right)\delta^2$,

$$x^2 + y^2 = \left(\frac{\partial^2 y}{\partial x^2}\right)\delta^2\left[i(d-\delta) + \frac{1}{4}\left(\frac{\partial^2 y}{\partial x^2}\right)\delta^2\right], \; x + d = \delta,$$

and in the limit, as δ vanishes,

$$\frac{\sqrt{x^2+y^2}}{x+d} = \sqrt{id\left(\frac{\partial^2 y}{\partial x^2}\right)},$$

i. e. is not 0. If the conic is a circle, $d = \infty$, and for every finite point $x + d$ is infinite and $\sqrt{x^2+y^2}$ is finite, so that the ratio in question is *constantly* 0.

Of course this is not the shortest proof of the last theorem. Indeed, the definitions of focus and directrix show that the equation of the conic, with the same choice of co-ordinates as before, will be of the form

$$x^2 + y^2 - e^2(x+d)^2 = 0,$$

where e is a constant, and we have directly, for any point (x, y) of the curve,

$$\frac{\sqrt{x^2+y^2}}{x+d} = \pm e.$$

For a circle $d = \infty$, $e = 0$, $de = r$, *where* r *is the radius.*

BALTIMORE, September, 1886.

*Klein's Ikosaeder.**

BY F. N. COLE, PH. D.

Prof. Klein's work on the Theory of the Ikosaedron is thoroughly representative of the characteristic tendencies of the German mathematical school of which Clebsch was the founder, and in which, since 1870, Klein has held a foremost place. The wonderful ability and success of Clebsch in giving to an algebraic problem a geometrical form, in replacing complicated relations of pure quantity by properties of arrangement of a geometrical configuration, constitute the chief charm of his works for the majority of his readers. The elegance of this geometrical-algebraical method appeals to and gratifies the artistic sense, while at the same time it gives the subject a broader sweep and a deeper reality. The practical demonstration of the possibility, already theoretically evident, of interweaving and unifying two fundamentally distinct branches of mathematics was particularly appropriate and welcome at a time when the rapid development of the modern theories had led to an extreme specialization of work and study. The evident necessity of a thorough and complete investigation of the relations and applications of the various mathematical theories to each other immediately opened up a new field of mathematical research, which has been abundantly productive, and bids fair not to be exhausted for many years to come.

Clebsch's work† in this direction concerned particularly the mutual relations between the theories of the Abelian integrals and of invariants and covariants and those of the corresponding geometrical configurations. Other investigators have examined the deeper algebraical-geometrical nature of the algebraic functions.‡ The geometrical side of the theory of invariants is perhaps more

* Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade, von Felix Klein. Leipzig, Teubner, 1884.

† An account of Clebsch's mathematical services, by his friends and pupils, appeared in the Math. Ann. Bd. VII.

‡ There belong here, first of all, the well-known theories of the representation of the algebraic functions by Riemann's surfaces, part of which, to be sure, are contemporary with Clebsch's work ; I refer, however, more especially to the investigations of which the article by Brill and Nöther. Math. Ann. Bd. VII, is a type.

conspicuous than the analytical, but it is very desirable that a purely geometrical theory of invariants, particularly of the binary forms, should be established in the way which Klein has indicated.* Of chief interest, however, for the present discussion, are the remarkable systems of relations which exist between the theory of *groups of operations*, of which the theory of substitutions constitutes a part, and nearly all other mathematical branches. It seems, indeed, as if the "Gruppentheorie" supplied, to a large extent, the essential formal structure to the various other theories, which then differ from each other only in the phase in which they are viewed. A characteristic feature of the modern mathematics is the predominant importance of theories, like that of groups of operations, which deal with discontinuous quantities. The theories which deal mainly with continuity have retreated decidedly into the background. It is a remarkable and suggestive fact that, scarcely two hundred years after the discovery of the Calculus, the higher mathematics has already exhibited a strong tendency to converge toward the oldest of all mathematical sciences, that of harmonious discontinuity—the theory of numbers.

The fundamental idea of Klein's entire mathematical work has been the investigation of a portion of this theory of operations which has a particular geometrical interest, while at the same time it is compactly united, through its analytical nature, with the entire Modern Algebra, using this name in its broadest sense. In the preface to the Ikosaedron, and in his "Vergleichende Betrachtungen über neuere geometrische Forschungen," will be found a statement by Klein himself of the programme which he had already developed as early as 1870, and which he has since accurately followed. It is the investigation of the groups of transformations of space into itself, and of those properties of space which remain unaltered—invariant—by these transformations, which has been the directing principle of his methods. Among these various transformations, those which are linear in the system of variables employed—the collineations—play a most important part. Analytically these are represented by

For space of 1 dimension
$$x_1' = \alpha x_1 + \beta x_2,$$
$$x_2' = \gamma x_1 + \delta x_2;$$

For space of 2 dimensions
$$x_1' = \alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3,$$
$$x_2' = \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3,$$
$$x_3' = \alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3;$$

* Eintritts-Programm at Erlangen: Vergleichende Betrachtungen über neuere geometrische Forschungen, 1872, Note VII at the end of the book. See also Lindemann's article in Math. Ann. Bd. VII: "Ueber die Darstellung binärer Formen," etc.

For space of 3 dimensions
$$\begin{aligned} x_1' &= \alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3 + \delta_1 x_4, \\ x_2' &= \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3 + \delta_2 x_4, \\ x_3' &= \alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3 + \delta_3 x_4, \\ x_4' &= \alpha_4 x_1 + \beta_4 x_2 + \gamma_4 x_3 + \delta_4 x_4. \end{aligned}$$

Evidently we are not limited to any number of dimensions, but may construct in general n linear equations between any two sets of n variables each. The connection of the theory of groups of transformations with that of the invariants of Cayley is now evident. The latter are unchanged by the infinite and continuous groups of linear substitutions just mentioned, namely, by *all* the collineations of the corresponding space. From this point of view, the theory of the Cayley invariants* appears as a grand division of this greater theory, which then includes in itself the entire geometry, not only the projective, but also (and this is, as Klein has indicated, an interesting development†) the metrical.

On the other hand, Klein has studied the *discontinuous linear groups*, both those which contain a finite, and those which contain an infinite number of operations. For those who are not acquainted with this theory, I may cite, as examples of these two species of groups, the rotations of any regular polyedron which bring it to a position congruent with its initial one, and the binary group of linear substitutions $\omega' = \alpha\omega_1 + \beta\omega_2$ which gives all the periods of an elliptic function when two independent ones, ω_1 and ω_2, are known. The latter contains an infinite number of operations, which, however, are discrete (α and β being integers). Included in this latter group is also the infinite group $\omega_1' = \alpha\omega_1 + \beta\omega_2$, $\omega_2' = \gamma\omega_1 + \delta\omega_2$ when $\alpha\delta - \beta\gamma = 1$.‡ In this review of the specific portion of Klein's work contained in his "Ikosaeder," it would evidently be out of place to give any detailed account of his work as a whole, but I have thought it desirable to give the preceding fragmentary sketch, in order that the "Ikosaeder" might appear in its proper position with respect to other portions of the more general theory. Having now developed so much of the latter as is necessary for my purpose, I proceed at once to the consideration of the ikosaedron itself.

The theory of the algebraic equations, from its central position in the

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* **Cayley invariants.** I have applied this name to the ordinary invariants in order to distinguish them from functions unchanged by other groups of transformations, to which the name invariant is equally applicable.

† See the Vergleichende Betrachtungen above, p. 12.

‡ See Briot and Bouquet's Theorie des Fonctions Elliptiques, p. 234.

modern mathematics, is peculiarly well adapted to serve as a base from which connection may be made with the various allied disciplines. The relation between it and the theory of substitutions is a perfect dualism, the propositions of the one being exact reproductions of those of the other; with the theory of invariants it stands by its very nature in the closest relation; and if we regard the coefficients in the equation as functions of one or more variables, the equation represents at once a geometrical configuration in space of one or more dimensions, while the discussion of the nature of the relation between the roots of the equation and its variable coefficients is the precise field of the theory of functions. It was therefore an excellent reason which led Klein, in writing a work in which the various branches of modern mathematics should be brought into a closer intimacy and mutual dependence, to select the theory of equations as a central feature. Not that this theory by any means assumes the greatest prominence in this work, but that in passing from one portion to another it always serves as a convenient stepping-stone, or a sort of central station. That exactly the equation of the fifth degree is chosen is not merely because we come upon it naturally after the solution of that of the fourth, but much more, from Klein's point of view, because the theories of the equations of the first five degrees constitute a closed system by themselves, the nature of which is most intimately connected, or rather identified, with the theory of the finite groups of linear transformations of a *single* variable. These groups present themselves, implicitly or explicitly, in every phase in which the theory of these equations can be studied, and are of such fundamental importance that Klein has preferred to devote the first portion of the work before us to a thorough exposition of their theory from all points of view, while the treatment of the general equations of the first five degrees appears in the second part as an extended application and development of this theory.

Among the various forms in which this theory of the finite linear groups of a single variable appears, the most tangible is undoubtedly that of the theory of those rotations of the ikosaedron and the kindred regular polyedra which bring these configurations to a final position which is congruent to the initial one. To this theory Klein has assigned the greatest prominence in the work before us, and he has made it the point of departure for the entire treatment of the subject. The first chapter of the book is occupied with a discussion of the groups of rotations of these regular bodies, while in the second chapter the immediate relation to the linear transformations of a single variable is exhibited by the introduction

of the stereographic projection of Riemann. Thus: the ikosaedron is supposed to be inscribed in a sphere. By projection of the ikosaedron edges from the centre of the sphere upon the spherical surface, we obtain on the latter a geometrical configuration which, for our purposes, completely replaces the ikosaedron itself. It is in the sense of this configuration, and not of the regular geometrical solid, that the name "Ikosaeder," as used by Klein, is to be understood. Obviously this results in no inconsistency with what here precedes, for all rotations which leave the ikosaedron congruent with its original position will do the same for the surface configuration. If now the sphere be placed on a horizontal plane, so that the point of tangency, which we may call the south pole of the sphere, is a vertex of the "Ikosaeder," and if then the entire configuration on the sphere be projected from the uppermost vertex, or north pole, on the plane, the resulting plane configuration furnishes valuable assistance in the further development of the theory.

To fix the position of a point on the sphere, we introduce a rectangular co-ordinate system with its origin at the centre of the sphere, and denote the co-ordinates of any point referred to this system by ξ, η, ζ. In the plane* we take another co-ordinate system, referred to two rectangular axes parallel respectively to the axes of ξ and η. The co-ordinates of any point in the plane being x and y, we denote the point by $z = x + yi$, this being the ordinary notation of the geometrical representation of complex numbers. Between the co-ordinates of any point on the sphere and those of its projection in the plane there exist the relations $x = \frac{\xi}{1-\zeta}$, $y = \frac{\eta}{1-\zeta}$, $x + iy = \frac{\xi + i\eta}{1-\zeta}$. If now the sphere undergo any rotation, including as a particular case those rotations which leave the ikosaedron congruent to its initial position, any point ξ, η, ζ will take a new position ξ', η', ζ' and at the same time its projected point z in the plane will become a new point z'. Now, we may suppose the whole of space to be rotated with the sphere, and such a rotation of space is a collineation, *i. e.* a transformation in which all points on a straight line become points on a straight line. Such transformations are denoted analytically by the system of linear equations of page 46. The effect of any rotation of the sphere is therefore to convert the point ξ, η, ζ into the point

$$\xi' = \frac{a_1\xi + b_1\eta + c_1\zeta}{d_1\xi + e_1\eta + f_1\zeta}, \quad \eta' = \frac{a_2\xi + b_2\eta + c_2\zeta}{d_1\xi + e_1\eta + f_1\zeta}, \quad \zeta' = \frac{a_3\xi + b_3\eta + c_3\zeta}{d_1\xi + d_2\eta + d_3\zeta}.$$

* The reader will notice that Klein supposes this second co-ordinate system to be projected on the sphere, thus dispensing with the plane. The "Ikosaeder" contains at the end a figure of the projection of the ikosaedron as the plane.

And now it appears that the corresponding transformation in the plane converts the point z into a new point z' which is related to the first point by the *linear* equation $z' = \frac{az+\beta}{\gamma z+\delta}$, where $\alpha, \beta, \gamma, \delta$ are constant quantities independent of z. This is a particular case of a more general proposition, the proof of which is not difficult. Thus there are in all ∞^{15} distinct collineations of space, namely, those which are defined by four equations of the form $x'_i = \alpha_i x_1 + \beta_i x_2 + \gamma_i x_3 + \delta_i x_4$. These contain $16 - 1$ essential constants. Of these, those which leave a given quadric surface (in the present case a sphere) unchanged, being limited by 9 conditions, the preservation of the nine coefficients in the equation of the surface, form ∞^6. The projection of these ∞^6 transformations of the spherical surface into itself gives ∞^6 transformations in the complex plane which are defined by $z' = \frac{az+\beta}{\gamma z+\delta}$, or by this equation together with the replacing of z by its conjugate value, which amounts to a reflection of the plane on its real axis.

Among these ∞^6 transformations of the spherical surface, we are more particularly concerned with those 120 rotations and reflections on the diametral planes which transform the ikosaedron into itself. The effect of such a rotation or reflection is to permute in a certain order all corresponding points of the ikosaedron, their projections in the plane of course undergoing the same permutations. Analytically this geometrical permutation of points is represented (in case of the rotations) by the replacement of the co-ordinate z of every point in the plane by a linear function of itself $\frac{az+\beta}{\gamma z+\delta}$. We have thus, parallel with our 60 interchanges of points, a group of 60 linear functions, these being of such a nature that if we select any one of them, and put for z successively the co-ordinates of any set of corresponding points, we get the same co-ordinates again, but in the permuted order. The completion of the theory requires the determination of these analytical expressions, which, however, need not be calculated separately, but are all obtainable by repetition and combination of two, the S and T of Klein, S denoting a rotation about the vertical axis through an angle $\frac{2\pi}{5}$, and T a rotation of 180° about the axis which bisects one of those ikosaedron edges which meet in the north pole of the sphere. For these we have* $S: z' = \varepsilon z$ and $T: z' = \frac{(\varepsilon - \varepsilon^4) z - (\varepsilon^2 - \varepsilon^3)}{(\varepsilon^2 - \varepsilon^3) z + (\varepsilon^4 - \varepsilon)} \left(\varepsilon = e^{\frac{2\pi i}{5}}\right)$. These 60 rotations combined with $z' = x' + y'i = \bar{z} = x - yi$ give the 60 reflections in addition.

* See Klein, p. 41.

The 60 ikosaedron rotations convert any point on the spherical surface into 60 similarly situated points, which then constitute a configuration which is unaltered, invariant, for this group of rotations. There are, however, certain of these invariant configurations which contain a less number of points. Thus the 12 vertices of the ikosaedron, the 20 centre-points of its faces and the 30 middle points of its sides are each configurations of the invariant character. To each invariant configuration belongs an invariant analytic function, namely, the analytical expression of the configuration, which is obtainable by multiplying together the values of the complex variable corresponding to the projections of the separate points of the configuration. Now, it appears that in the present case every such invariant function can be composed rationally from those three which represent the three special cases mentioned above of the ikosaedron vertices, etc., the T, H, f, of the book,* and from this it follows easily that for different values of Z, $Z : Z - 1 : 1 = H^3 : - T^2 : 1728f^5$ defines any one of the groups of 60 points which are related to each other in the way we have considered. This equation, which is of degree 60 in $\frac{z_1}{z_2}$, contains in itself implicitly the general equation of the fifth degree, for it is a resolvent of the latter. If for a given value of Z we can determine the corresponding group of 60 correlated points on the sphere, we can at once solve the equation of the fifth degree, and *vice versa.*

In the "extended group" of 120 transformations, composed of the 60 rotations above and the 60 possible reflections on planes of symmetry, the former constitute a sub-group by themselves, while the latter do not, since two reflections give a rotation, not a reflection. Moreover, if we take successively a reflection, a rotation, and the same reflection reversed, we get a rotation. *The group of the rotations is self-conjugate*† within the extended group.

In these first two chapters of the book will be found a very complete and elegant account of the theories, not only of the ikosaedron, which is identified with the theory of the equation of the fifth degree, but also of the oktaedron, the tetraedron and the "diedron," whose theories are of similar importance for the equations of the fourth and third degrees and the cyclical equations respectively. The geometrical interpretation which this theory gives to a portion of

* See pages 55–61.

† I use "self-conjugate sub-group" in translating Klein's "ausgezeichnete Untergruppe" and Jordan's "groupe permutable."

the theory of substitutions is well worthy of being studied, especially the geometric conception of the position of a self-conjugate sub-group within the entire group, and the resulting geometrical demonstration that the oktaedron group contains such a sub-group, while the ikosaedron group does not.

Before leaving this part of the book, one more matter deserves to be noted. If we make our 60 linear substitutions in the plane, corresponding to the 60 rotations of the sphere, homogeneous by introducing for z, $\frac{z_1}{z_2}$, writing $\begin{matrix} Z_1'=\alpha z_1+\beta z_2, \\ Z_2'=\gamma z_1+\delta z_2 \end{matrix}$, it appears that the "identical rotation" $z_1'=-z_1$, $z_2'=-z_2$, will appear among the homogeneous substitutions, so that there will be in all $120=2.60$ of these. The relation between the homogeneous and the non-homogeneous group is therefore a hemiedric isomorphism. This is not to be avoided. There is no group of binary linear substitutions which is holoedrically isomorphic with the non-homogeneous group considered. This is a fact of considerable importance for the theory of the equation of the fifth degree, as is seen later on. On this depends the appearance of an "accessory" square root in the solution.

This theory of the correspondence between groups of linear transformations of a single variable and groups of rotations of the regular polyedra receives a remarkable completion in the fifth chapter of the book. We have seen that to every group of rotations corresponds a configuration of points, to which configuration an invariant then belongs. If any point of the configuration be given, all others proceed from it by the rotations, or, if we consider the projection on the complex plane, by linear transformations, so that the entire configuration, and consequently its invariant, are at once known. The inverse of this problem would be, given the invariant Z, to find the various points z of the configuration. Only one solution of this inverse problem is necessary; *i. e.* geometrically we need find only one point of the configuration, since all others are then obtainable by known linear transformations from this one. Leaving the actual solution of this problem out of consideration for the time being, we may attempt to determine *à priori* all possible problems whose different solutions possess this remarkable relation to each other. And now it appears that the choice of the groups of rotations of the regular polyedra as the object of the present investigation was no arbitrary one, but that these rotations and the corresponding groups of linear transformations of a single variable constitute a closed system by themselves, that, namely, every problem whose different solutions all proceed from any one of them by linear transformations is identical with one of the

problems of this system. In other words, every algebraic equation whose roots are all linear functions of any one of them is either an ikosaedron, an oktaedron, a tetraedron or a cyclical equation, or can be reduced to one of these by replacing the invariant Z by a linear function of itself, $Z' = \frac{aZ+b}{cZ+d}$, and z by $z' = \frac{\alpha z+\beta}{\gamma z+\delta}$. Or, again, every finite group of linear transformations of a single variable is holoedrically isomorphic, *i. e.* formally identical, with some one of the groups of rotations of the regular polyedra. This is the centre-point of this entire theory. *The problem with the various phases of which the present work deals is in no way an arbitrary one, but constitutes, in all its developments, a complete whole, perfectly defined in every direction by conditions and limitations inherent in its own nature.* At the same time, these conditions and limitations serve exactly to characterize the position of the present theory with respect to other related or more comprehensive theories. Thus, the finite ternary, quaternary, etc., linear groups admit of an analogous geometrical treatment; or, instead of increasing the number of variables, we may consider binary groups containing an infinite number of operations. On the other hand, I have already stated that the theory of the general algebraic equations of the first five degrees is identical with that of the rotations of the various regular polyedra and of the finite binary linear groups. Similarly, we might seek for corresponding relations between the theory of the higher equations and that of the ternary, quaternary, etc., groups. In this theory, which was proposed by Klein,* considerable progress has already been made. Thus, for the general equation of the sixth degree, the quaternary group of the transformations of the Borchardt moduli $\xi, \eta, \zeta, \vartheta$, and, for certain equations of the seventh and eleventh degrees, the ternary and quinary linear groups belonging to certain functions proposed by Klein,† play the same part as the groups of a single variable in the present case. The theory of the infinite groups leads to interesting developments, which will be considered later.

The problem with which the book before us has to deal is now completely defined on all sides. On the one hand, it is the examination of all groups of linear transformations of a single variable, which then, as we have just seen, is

* See in particular the article, "Ueber die Auflösung gewisser Gleichungen vom siebenten und achten Grade," Math. Ann. XV.

† *Ibidem.* See further the article in the same volume, "Ueber die Transformation elfter Ordnung," etc.

essentially identical with the determination of all groups of rotations of regular polyedra. This part of the theory, therefore, enjoys the joint benefit of the analytical and geometrical methods from the start. Having, through this examination obtained a complete understanding of the problem itself, we have, on the other hand, to determine in detail all other problems which are equivalent in their nature to the one considered. The development of the former of these two divisions of the subject leads at once to the consideration of the analytical nature of the actual solution of our problems in the third chapter, while the fourth chapter deals with the Galois theory of substitutions, to which the theory of mutually equivalent algebraic problems naturally belongs.

The third chapter is particularly instructive, on account not only of the insight which it gives into the nature of the actual solution of the problem and of the resulting extension of our knowledge of the theory of functions, but also because of the fundamental connection here exhibited between the theory of these new functions and the theory of the linear differential equations of the second and third orders. The introduction of these equations adds a new and most valuable instrument of research to those already at our disposal, and at the same time the dominant purpose of the book, the extension of the subject to meet all other related branches, is fully carried out in this direction.

The introduction of the differential equations into this theory is accomplished in a remarkably natural manner. We have a certain analytic or geometric configuration to which an invariant Z belongs, the elements of the configuration being denoted by η. One of these last being given, say η, all others are determined by $\xi_1 = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}$, where α, β, γ, δ are known quantities. If we differentiate this equation three times with respect to Z, and eliminate α, β, γ, δ from the resulting and original equations, we shall have a differential expression which, being independent of α, β, γ, δ, is independent of linear transformation. This expression is the equation $\frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2 = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2$, of which either side is such a "differential invariant." This Schwarzian derivative, as Cayley calls it, is therefore independent of the separate elements of the configuration, and depends only on the configuration as a whole; *i. e.* it is symmetrical in the roots of the corresponding equation, and is therefore a rational function of the invariant Z. The determination of the nature of this rational function is accomplished easily by aid of considerations from the theory of functions based on the

geometry itself. The differential equation then assumes the form

$$[\eta]_z = \frac{\nu_1^2-1}{2\nu_1^2(z-1)^2} + \frac{\nu_2^2-1}{2\nu_2^2 z^2} + \frac{\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} - \frac{1}{\nu_3^2} - 1}{2(z-1)z},$$

where $[\eta]_z = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$, and ν_1, ν_2, ν_3 are characteristic numbers belonging to the special configurations already mentioned above, and denote in each case the number of points in the configuration which coincide in sets for the corresponding special values of Z. Thus, for the ikosaedron, $\nu_3 = 5$, corresponding to the five faces of the ikosaedron which meet at each vertex, and to the 5 points which consequently coincide when Z has the value belonging to the configuration of the vertices.

With this introduction of the differential equation of the third order, this portion of the theory is by no means complete. There remains to be noticed an important connection between the theory of this differential equation of the third order and those linear differential equations of the second order whose coefficients are rational functions of the independent variable. Thus if $y'' + py' + qy = 0$ be such an equation, and if we form the ratio of two particular solutions $\frac{y_1}{y_2}$, then, if the independent variable Z describe a closed path on its Riemann's surface, $\eta = \frac{y_1}{y_2}$ will become $\xi = \frac{\alpha y_1 + \beta y_2}{\gamma y_1 + \delta y_2}$. On account of this linear relation η satisfies a differential equation of the third order, whose left-hand side is identical with that of the equation already considered, while its right-hand side is a rational function of Z. It appears, furthermore, that not only can we always obtain from a linear differential equation of the second order another of the third order, but that we can also always accomplish the solution of the inverse problem. There is therefore a linear differential equation of the second order belonging to that of the third order which we have deduced from the theory of the linear groups. The importance of this equation of the second order depends on the fact that its solution is a special case of the Riemann function P, the connection of which with the hypergeometric series of Gauss is well known.

Returning to the differential equation of the third order, we have already seen how our problem admits of extension by the introduction of infinite groups of linear transformations of a single variable. In the fifth chapter will be found an account of the connection of this theory with that of the functions

of Schwarz. Thus, using the notation already explained, $\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3}$ is always greater than 1 for all the problems which the present work treats. If now we give to ν_1, ν_2, ν_3 any integral values, we get a series of new functions. The geometrical representation of these on the sphere is very interesting, and is given in some detail, but of especial importance is the system of functions obtained by putting $\nu_1 = 2$, $\nu_2 = 3$, and giving ν_3 successively all integral values from 2 on. This gives in succession for $\nu_3 = 2, 3, 4, 5$ the diedron, the tetraedron, oktaedron and ikosaedron, and then an infinite series of transcendental functions ending with the elliptic modular functions for $\nu_3 = \infty$. Every one of these Schwarz functions, ν_1, ν_2, ν_3, is a rational function of every other, ν_1', ν_2', ν_3, if ν_1', ν_2', ν_3 are integral multiples of ν_1, ν_2, ν_3. For the present case, where $\nu_1 = 2$, $\nu_2 = 3$, every function of this series will therefore evidently be a rational function of $\nu_1 = 2$, $\nu_2 = 3$, $\nu_3 = \infty$. It is exactly for this reason that the equation of the fifth degree, among others, can be solved by aid of the elliptic modular functions. By solution of such an equation, we mean simply the representation of its roots as rational functions of known quantities, in this case, of the known elliptic transcendents.

The actual solution of the problem having been thus discussed, and its bearing on other mathematical branches having been fully treated, it remains to determine the nature of all equivalent problems, *i. e.* of all problems whose solution is implicitly involved in that of the present one. For this purpose the Galois theory of substitutions is the efficient instrument, while it also secures us a completer knowledge of the essential nature of the internal structure of the problem. A development of this theory in its more essential and pertinent features occupies the fourth chapter of the book.* Starting from any algebraic equation of degree n, we may construct rational functions of its n roots $x_1 \ldots . x_n$, and consider the effect which a permutation of these roots has on the form and value of these functions. A function may remain unchanged for all permutations of the roots, and is then symmetrical, or it may be changed by certain permutations and unchanged by others; and the number of permutations which leave it unchanged may vary from $n!$ in the case of the symmetrical functions to 0 for the utterly unsymmetrical ones such as $a_1x_1 + a_2x_2 + \ldots . + a_nx_n$, where $a_1 \ldots . a_n$ are all

* Here the reader will do well to take a course of parallel reading in the recent work of Netto mentioned in the footnotes of the Ikosaeder: "Substitutionentheorie und ihre Anwendungen auf die Algebra." Teubner, Leipzig.

different constants. All those permutations which leave any function unchanged constitute a group, since evidently, if the permutations σ_i and σ_j leave a function unchanged, then their successive application, denoted by $\sigma_i\sigma_j$, will also leave it unchanged. To every function then corresponds a group of substitutions, and, conversely, we can always find for every group a function which shall be unchanged by its operations. All functions belonging* to the same group constitute a family—Gattung†—which possesses the property that every function of the family is a rational function of every other one, with coefficients which are rational in the coefficient of the original equation. If we consider any function ϕ, and determine the corresponding group of substitutions, $1, \sigma_1 \ldots . \sigma_{r-1}$, then, if we apply any other substitution σ_i to ϕ, we shall get a new function, ϕ_2. Thus, $\sigma_i\phi_1 = \phi_2$. ϕ_2 is called conjugate to ϕ_1. Evidently there can be only a finite number of functions conjugate to a given one. If k be the number of values which a function takes when subjected to all possible permutations of the roots, and if r be the order of the group belonging to the function, *i. e.* the number of operations included in it, then $rk = n!$ Both r and k must therefore be factors of $n!$, which greatly restricts the number of possible groups and values of functions.

If $\phi_1 \ldots . \phi_k$ be all the values of a given function, then $\Sigma\phi_i$, $\Sigma\phi_i\phi_j$, etc., will all be symmetrical functions of the roots, and therefore rational functions of the coefficients of the original equation. We may therefore obtain an equation of degree k of which the coefficients are "rationally known" quantities, and of which $\phi \ldots . \phi_k$ are the roots. Such an equation is a *resolvent* of the original equation. If the group of ϕ_1 be $1, \sigma_i \ldots . \sigma_{r-1}$, and if σ_i converts ϕ_1 into ϕ_2, then the group of ϕ_2 will be $1 = \sigma_i 1, \sigma_i^{-1}, \sigma_i\sigma_1\sigma_i^{-1}, \ldots . \sigma_i\sigma_{r-1}\sigma_i^{-1}$. Here two important cases must be distinguished. Either the groups of $\phi_1 \ldots . \phi_k$ are distinct, or they coincide. If they are distinct, not only are the ϕ's rational functions of the x's by definition, but the x's are also rational functions of the ϕ's, since every permutation of the x's permutes the ϕ's also; so that $\alpha_1\phi_1 + \alpha_2\phi_2 + \ldots . + \alpha_k\phi_k$ is a $n!$ valued function, and therefore every other rational function of the roots is a rational function of this. In this case, if we can solve the equation for the ϕ's, that of the x's is likewise solved, and *vice versa*, if the x equation is solved, the ϕ equation is solved with it; *i. e.* the two involve only the same irrationalities. But if the groups of the

* Any function unchanged by the operations of the group is said to *belong* to the group.

† See Kronecker's Festschrift, Crelle 92, for a further development of this idea. Cf. also in this connection Bachmann: Ueber Galois' Theorie der algebraischen Gleichungen, Math. Ann. XVIII.

$\phi_1 \ldots . \phi_k$ are coincident, in which case the group is said to be self-conjugate, the x's will not be rational functions of the ϕ's, and consequently the solution of the ϕ equation will not involve that of the x equation, but we still have to solve a second equation to determine the x's from the ϕ's. To compensate this, the ϕ equation is itself easier to solve, because of the relations which exist between its roots, every root being a rational function of every other root. In the former case, the given x equation and its resolvent present exactly the same problem; in the latter case, the introduction of the resolvent decomposes the problem into two simpler steps. The possibility of such a reduction depends, therefore, on the presence of a self-conjugate group. Such a group occurs for the oktaedron, and is characteristic of the equation of the fourth degree. For the fifth degree and the ikosaedron, on the contrary, no such group exists.

Through these considerations the question of the nature and of the determination of those problems which are equivalent to the given one may be regarded as settled. Equivalent algebraic problems are those which are resolvents of each other. To this idea of resolvent equations must be added that of the Galois group of an equation, when the theory will be complete in this direction. The notion of the group of an equation is essentially identical with the conception of what, in the problem of its solution, is to be looked upon as known or given, *i. e.* the *Rationalitätsbereich* of Kronecker. If the general equation of any degree be proposed for solution, evidently all that is known is the coefficients, and the solution of the equation consists in determining the roots in terms of these; but for special equations we may know other rational relations between the roots besides the symmetric ones. For instance, for the cyclical equation $x_n = 1$, where n is a prime number, we know that every root is a power of every other one except 1. Again, for the ikosaedron equation, we know that every root is a rational function of every other one. If, now, we construct every rational relation which exists between the roots in the form of an equation whose right-hand side is 0, the Galois group of the given equation is that group of permutations of the roots which leaves all these relations unchanged in value.*

The immediate application of this to the ikosaedron, oktaedron, etc., equations is obvious. In each case the group of the equation is composed of those

*I wish to append here a very concise definition of the Galois group of an equation given by Prof. Klein in one of his lectures: "The Galois group of an equation is the group of permutations of its roots which possesses the two properties—1st, that all corresponding rational functions of the roots are rationally known, and 2d, that it is the smallest group for which this is true."

permutations of the roots, or points of the configuration, which are produced by all the rotations of the corresponding polyedron. Thus the group of the ikosaedron equation is composed of 60 permutations of its roots, out of a possible 60!

The connection between this theory of the group of an equation and that of resolvents of the equation is complete when we notice that all resolvents of any equation have precisely the same group with the original equation, unless indeed the appearance of a self-conjugate sub-group should interfere with this. For, excluding this possibility, every permutation of the x's will produce a permutation of the ϕ's, so that, corresponding to the group of the x equation, there will exist a group of permutations of the ϕ's, which must then be the group of the ϕ equation. For every function which is rational in the x's is also rational in the ϕ's, and *vice versa*, and every function which, regarded as a function of the x's, is unchanged by a group of x permutations, will be, regarded as a function of the ϕ's, unchanged by the corresponding ϕ permutations.

All problems, therefore, equivalent to the given ones are resolvents of these, whereby they all possess a common characteristic—they all have the same group. To finish this portion of the theory, it only remains to construct all such resolvents. And here the actual connection of the ikosaedron and the other polyedra with the equations of the first five degrees, etc., is made obvious. *The general equations of degrees 2–5 and the cyclical equations are resolvents of the ikosaedron, etc., equations.* Thus the 60 rotations of the ikosaedron and the 60 even permutations of the roots of the general equations of the fifth degree are symbolically identical, *i. e.* holoedrically isomorphic; and similar relations hold for the simpler cases.

In the fourth chapter of the book, the more important resolvents are actually obtained. Among them, that of the sixth degree and the *Hauptgleichung* of the fifth degree are particularly to be noticed. The former is the equation for the transformation of the fifth degree of the elliptic functions. The latter will play an important part in the second portion of the book.

Within the brief space of a review, it is of course impossible even to touch upon many matters of great interest, for which reference must be made to the reviewed work itself. And having now developed the fundamental ideas of the first part of the "Ikosaeder" in considerable detail, I shall be obliged to forego giving any adequate account of the many important and elegant theories contained in the second part of the book. This, however, will by no means imply that this portion of the work is of less importance than the other; on the contrary, it

contains some of the most valuable features of the entire book. But, on the one hand, the structural nature of the problems dealt with here is already implicitly treated in the first part, and, on the other hand, the further developments which this last half of the book contains would scarcely be comprehensible without the actual study of the book itself. No account of the "Ikosaeder" would, however, be complete which did not call attention to the historical development of the theory of the equations of the fifth degree contained in the opening chapter of the second part, which may well be read before any other portion of the book. The abundant historical matter contained here, and the profusion of foot-notes throughout, are most valuable features, and constitute in themselves a complete encyclopædia of information. The methods of Tschirnhaus, Bring and Jerrard, and later those of Jacobi, Hermite, Brioschi and Kronecker, are all given in short sketches, with full references to the original works, which will materially assist the student in following the historical and philosophical development of these important systems. The reader will continually find himself referring to this, and to the fifth chapter of the first part, for the general direction and tendency of the broader phases of the theory.

Of the later chapters of the book, the fourth may be especially noted. It contains the treatment of the Jacobi modular equation of the sixth degree, which connects the work of Klein with the earlier theory of Kronecker, Brioschi, Hermite and others. I can only mention here that this theory is closely connected with that of a ternary group of linear transformations, just as the ikosaedron problem is related to that of the binary linear group of 60 transformations.

Before concluding, it remains to mention one matter which is of considerable general importance, and is characteristic of Klein's entire method. It is the exact meaning of the phrase "solution of an equation" in Klein's sense of the word. It will already be evident that this is something very different from the common conception of the words. Thus it is ordinarily said that the general equation of the fifth degree is solved by aid of the elliptic modular functions, whereas, from Klein's point of view, the introduction of these transcendental irrationalities is in no way essential to the theory, in fact, rather lies outside the region of the present work. According to Klein, an equation is to be regarded as solved when its complete *structural nature* is fully known. This includes the knowledge of the nature of all connection between the roots, all relations between different resolvent functions, all functional quantities which may be

regarded as known; in short, the entire Galois theory as applied to the case in hand; further, the nature of all those properties of the problem which are unchanged by groups of operations of any character, particularly those invariants which belong to groups of linear transformations; again, the complete knowledge of the nature of the actual solution as based on the method of the differential equations, which last then creates a new portion of the theory of functions; and finally, an adequate geometrical or hypergeometrical representation shall be found for all these characteristic properties. It is with the *implicit* nature of the equation that this theory deals, while the explicit form of the actual solution is a matter of comparative unimportance. The introduction of the elliptic transcendents in the solution of the equation of the fifth degree appears from this point of view, like the introduction of the trigonometric functions in the solution of the equation of the third degree, to belong rather to the theory of transcendental quantities than to the theory of equations.

Starting out from this broad conception, Klein has proposed a general theory of equations which shall contain in itself these various treatments. Thus, if we have to solve an equation, we first of all determine its Galois group. This having been done, the next step will be in each case to determine a finite group of linear transformations, of as few elements as possible, which shall be holoedrically isomorphic with the Galois group of the equation. The interpretation of these linear transformations as collineations of the corresponding space, and the determination of a corresponding invariant configuration in this space, form the basis of the geometrical treatment. Finally, a system of differential equations is to be obtained which are satisfied by the actual solution of the problem. The hyperelliptic functions may then be introduced as accessory irrationalities, just as the elliptic and trigonometric functions appear in the present theory.

CAMBRIDGE, September, 1886.

Wave Motion in Hydrodynamics.

By A. G. Greenhill, *Woolwich, England.*

The mathematical subject of Hydrodynamics is still, in some respects, in an elementary stage, insomuch as every fresh problem solved constitutes a distinct advance of the subject.

One of the most important applications of the theory of Hydrodynamics is to the question of the motion of Waves under gravity and other causes, and as the investigations on this subject are for the most part scattered about in various scientific periodicals, I propose in this article to collect together the chief results hitherto obtained, and to give also a general connected account of the mathematical theory, at the same time attempting to develop it in some directions.

In the mathematical treatment of Wave Motion we are constrained at present to employ the approximation of supposing the velocities of the liquid particles due to the wave motion to be sufficiently small for the squares, etc., of the particle velocities to be neglected; although it is singular that this approximation is not required in the first problem of wave motion ever solved, discovered by Gerstner in 1802, and afterwards independently by Rankine in 1862 (Stokes, Mathematical and Physical Papers, I, p. 219).

A list of the principal papers on the subject of Wave Motion and of their authors will be found in the Report on Recent Progress in Hydrodynamics, by W. M. Hicks, F. R. S., presented to the British Association.

1. The most convenient order to employ in the mathematical treatment of a problem in the subject of Wave Motion is: (I) The determination of the velocity function ϕ, or stream function ψ, satisfying the equation of continuity; (II) The determination of the boundary conditions to be satisfied at the sides of the containing vessel; (III) The most difficult part, the determination of the conditions to be satisfied in order that the free surface should be a surface of equal pressure, or, more generally, at the surface of separation of two liquids there should be no discontinuity of pressure.

To secure uniformity of notation in the treatment of waves under gravity, we shall suppose the co-ordinate axis Oz drawn vertically upwards, and the plane xOy taken generally in the undisturbed horizontal plane of the surface of separation of two liquids at which the wave motion is apparent, and then the axis Ox will be taken in the direction of propagation of the waves when straight-crested, and, therefore, perpendicular to the crests.

Then, to determine the wave motion in still water (still except for the slight disturbance of the wave motion), we must determine a velocity function ϕ. (I) Satisfying the equation of continuity for liquids:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0.$$

(II) Satisfying the condition

$$\frac{\partial\phi}{\partial\nu}=0$$

at a fixed boundary, $\partial\nu$ denoting an element of the outward-drawn normal of the boundary, or, more generally,

$$\frac{\partial\phi}{\partial\nu}=n,$$

the normal component velocity of the boundary, when movable. (III) Satisfying at the surface $z=0$, supposed a surface of equal or of no discontinuity of pressure, the dynamical equation

$$\frac{p}{\rho}+gz+\frac{\partial\phi}{\partial t}=H,$$

a constant, neglecting the squares of the velocities of the liquid particles.

At a free surface p is constant, and, therefore, $\dfrac{\partial p}{\partial t}=0$, so that, η denoting the elevation of the free surface,

$$g\frac{\partial\eta}{\partial t}+\frac{\partial^2\phi}{\partial t^2}=0.$$

But
$$\frac{\partial\eta}{\partial t}=\frac{\partial\phi}{\partial z},$$

and
$$\frac{\partial^2\phi}{\partial t^2}=-\frac{g}{l}\phi,$$

where l denotes the length of the equivalent simple pendulum of the wave motion, so that
$$l\frac{\partial\phi}{\partial z}=\phi;$$

at the free surface $z=0$, or $z=h$, some constant.

2. When surface tension T is taken into account, this equation must be replaced by

$$l\frac{\partial\phi}{\partial z}+\frac{Tl}{g\rho}\frac{\partial^3\phi}{\partial z^3}=\phi,$$

an equation due to Kolacek (*vide* Fortschritte der Mathematik, 1878). For, if ∂p denotes the excess of pressure in the liquid just below the capillary film over the external pressure above,

$$\frac{\partial p}{\rho}+g\eta+\frac{\partial\phi}{\partial t}=0.$$

But r and r', denoting the radii of curvature of any two vertical sections of the free surfaces by perpendicular planes,

$$-\partial p=T\left(\frac{1}{r}+\frac{1}{r'}\right)=T\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)$$

to one order of approximation, so that

$$T\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)=g\rho\eta+\rho\frac{\partial\phi}{\partial t};$$

and differentiating with respect to t, and replacing $\frac{d\eta}{dt}$ by $\frac{\partial\phi}{\partial z}$, remembering also that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-\frac{\partial^2\phi}{\partial z^2},$$

then

$$-T\frac{\partial^3\phi}{\partial z^3}=g\rho\frac{\partial\phi}{\partial z}-\frac{g\rho}{l}\phi,$$

or

$$l\frac{\partial\phi}{\partial z}+\frac{Tl}{g\rho}\frac{\partial^3\phi}{\partial z^3}=\phi.$$

3. *Digression on the Hyperbolic Functions.*

In the course of our investigations we shall require certain functions, called Hyperbolic Functions, from their connection with the hyperbola, which are analogous to the functions of the circle defined in ordinary trigonometry. As these functions are not defined and explained in all the ordinary text-books, we shall, for convenience, proceed to do so as follows:

(I) $\frac{1}{2}(e^v+e^{-v})$ is called the *hyperbolic cosine* of v, and is denoted by $\cosh v$.

(II) $\frac{1}{2}(e^v-e^{-v})$ is called the *hyperbolic sine* of v, and is denoted by $\sinh v$.

(III) $\frac{e^v-e^{-v}}{e^v+e^{-v}}=\frac{\sinh v}{\cosh v}$ is called the *hyperbolic tangent* of v, and is denoted by $\tanh v$; and so on, by analogy, with the rest of the circular functions.

From the *exponential* values of the cosine and sine, viz.,

$$\cos u = \frac{1}{2}(e^{iu} + e^{-iu}),\ \sin u = \frac{1}{2i}(e^{iu} - e^{-iu}),$$

when i denotes $\sqrt{-1}$, we see, by putting $u = iv$, that $\cos iv = \cosh v$, $\sin iv = i \sinh v$, $\tan iv = i \tanh v$, etc.; also,

$$\cos(u + iv) = \cos u \cosh v - i \sin u \sinh v,$$
$$\sin(u + iv) = \sin u \cosh v + i \cos u \sinh v, \text{ etc.,}$$

formulæ of great use hereafter. Therefore, also,

$$\cosh(u + iv) = \cosh u \cos v - i \sinh u \sin v,$$
$$\sinh(u + iv) = \sinh u \cos v - i \cosh u \sin v.$$

Analogous to the ordinary formulæ of circular trigonometry, we have

$$\cosh(\alpha + \beta) = \cosh\alpha \cosh\beta + \sinh\alpha \sinh\beta,$$
$$\sinh(\alpha + \beta) = \sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta,$$
$$\tanh(\alpha + \beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha \tanh\beta},$$
$$\sinh\gamma + \sinh\delta = 2 \sinh\frac{1}{2}(\gamma + \delta) \cosh\frac{1}{2}(\gamma - \delta),$$
$$\sinh\gamma - \sinh\delta = 2 \cosh\frac{1}{2}(\gamma + \delta) \sinh\frac{1}{2}(\gamma - \delta),$$
$$\cosh\gamma + \cosh\delta = 2 \cosh\frac{1}{2}(\gamma + \delta) \cosh\frac{1}{2}(\gamma - \delta),$$
$$\cosh\gamma - \cosh\delta = 2 \sinh\frac{1}{2}(\gamma + \delta) \sinh\frac{1}{2}(\gamma - \delta),$$

and so on.

4. *Waves in Still Water of Uniform Depth.*

Supposing straight-crested waves of length λ propagated in the direction of the axis of x with velocity U, we may begin by supposing the velocity function

$$\phi = f(z) \cos(mx - nt),$$

where $m = 2\pi/\lambda$, $n = 2\pi U/\lambda$, and $n^2 = g/l$, l denoting the length of the equivalent simple pendulum.

Then, from the equation of continuity,

$$\frac{d^2 f}{dz^2} - m^2 f = 0,$$

the solution of which is

$$f(z) = ae^{mz} + be^{-mz};$$

or, using hyperbolic functions,

$$f(z) = P \cosh mz + Q \sinh mz;$$

or, subject to the condition that

$$\frac{\partial \phi}{\partial z} = 0, \text{ when } z = -h,$$

h denoting the depth of the water,

$$f(z) = A \cosh m(z+h);$$

so that

$$\phi = A \cosh m(z+h) \cos(mx - nt).$$

Then, at the free surface $z = 0$,

$$l \frac{\partial \phi}{\partial z} = \phi;$$

or,

$$ml \sinh mh = \cosh mh;$$

or,

$$ml = \coth mh.$$

Then,

$$U^2 = \frac{n^2}{m^2} = \frac{g}{m^2 l}$$

$$= \frac{g}{m} \tanh ml = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda},$$

the well-known expression for the wave velocity.

When h/λ is small, we can replace $\tanh(2\pi h/\lambda)$ by $2\pi h/\lambda$, and then

$$U^2 = gh,$$

Kelland, Scott, Russell and Green's expressions for the wave velocity when the wave length is great compared with the depth of water.

When h/λ is large, we can replace $\tanh(2\pi h/\lambda)$ by unity, and then

$$U^2 = g\lambda/2\pi,$$

agreeing with Gerstner's and Rankine's expressions for the wave velocity in water of great depth.

5. Next, suppose there is a surface tension T at the free surface; then the condition

$$l \frac{\partial \phi}{\partial z} + \frac{Tl}{g\rho} \frac{\partial^3 \phi}{\partial z^3} = \phi,$$

when $z = 0$, leads to the relation

$$ml \sinh mh + \frac{Tl}{g\rho} m^3 \sinh mh = \cosh mh;$$

or,

$$U^2 = \frac{n^2}{m^2}$$

$$= g\left(\frac{1}{m} + \frac{Tm}{g\rho}\right) \tanh mh$$

$$= \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda}\right) \tanh \frac{2\pi h}{\lambda},$$

the general expression for the wave velocity under gravity and surface tension combined.

When λ is large or T small, we may put

$$U^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda},$$

as above; but when λ is small, the term $g\lambda/2\pi$ is insensible, and we can put

$$U^2 = \frac{2\pi T}{\rho\lambda} \tanh \frac{2\pi h}{\lambda},$$

the velocity of propagation of ripples of wave length λ due to surface tension T.

Supposing the depth of water h sufficiently large for $\tanh (2\pi h/\lambda)$ to be replaced by unity, then

$$U^2 = \frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda} = 2\sqrt{\frac{gT}{\rho}} + \left(\sqrt{\frac{g\lambda}{2\pi}} - \sqrt{\frac{2\pi T}{\rho\lambda}}\right)^2,$$

so that the minimum value of U is

$$\sqrt{\left(2\sqrt{\frac{gT}{\rho}}\right)},$$

and the $\lambda = 2\pi\sqrt{(T/g\rho)}$.

Sir W. Thomson proposes to distinguish by the name of *ripples* those waves whose length is less than the above critical value of λ (Phil. Mag. (4) xlii).

6. A slight extension of this problem may be made by supposing the capillary film of the surface to be replaced by a flexible cloth of uniform tension T and uniform superficial density σ resting on the surface of the liquid.

Then, assuming for the liquid motion a velocity function

$$\phi = A \cosh m(z+h) \cos(mx - nt),$$

as before, supposing A a small constant factor, and denoting by η the elevation of the surface, then, when $z = 0$,

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = mA \sinh mh \cos(mx - nt),$$

and, therefore,

$$\eta = -\frac{m}{n} A \sinh mh \sin(mx - nt).$$

Denoting by ∂p the excess of pressure just below the cloth over the atmospheric pressure above, then

$$\frac{\partial p}{\rho} + g\eta + \frac{\partial\phi}{dt} = 0;$$

or,

$$\partial p - g\rho \frac{m}{n} nA \sinh mh \sin(mx - nt) + nA \cosh mh \sin(mx - nt) = 0.$$

But the equation of motion of the cloth is

$$\sigma \frac{d^2\eta}{dt^2} = T\frac{d^2\eta}{dx^2} + \partial p;$$

so that, dropping the common factor $A \sin (mx - nt)$,

$$\sigma mn \sinh mh = T\frac{m^3}{n} \sinh mh + g\rho\frac{m}{n} \sinh mh - \rho n \cosh mh;$$

or,

$$U^2 = \frac{n^2}{m^2} = \frac{g\rho/m + Tm}{\rho \coth mh + \sigma m},$$

giving U the velocity of propagation of the waves, and reducing, when $\sigma = 0$, to the preceding case of a capillary film.

7. *Waves in Ice of Uniform Thickness Resting on Water of Uniform Depth.*

If the water is covered with ice, then the equation of vibration of the surface must be replaced by

$$\sigma \frac{d^2\eta}{dt^2} = -L\frac{d^4\eta}{dx^4} + \partial p,$$

where L denotes the flexural rigidity of the ice, the vibrations being now of the nature called *lateral* vibrations (Rayleigh, Theory of Sound, I, Chapter 8, §163), the *inertia* of each vertical section of the ice being supposed concentrated at the centre.

Then if e denotes the thickness of the ice, and E Young's modulus of elasticity,

$$L = \frac{1}{12} e^3 E, \text{ and } \sigma = e\rho,$$

supposing the ice of the same density as water, so that now

$$\sigma mn \sinh mh = -L\frac{m^5}{n} \sinh mh + g\rho\frac{m}{n} \sinh mh - \rho n \cosh mh;$$

or,

$$U^2 = \frac{n^2}{m^2} = \frac{g\rho/m + Lm^3}{\rho \coth mh + \sigma m}$$

$$= \frac{g\lambda/2\pi + \frac{2}{3}\pi^3 e^3 E/\rho\lambda^3}{\coth(2\pi h/\lambda) + 2\pi e/\lambda},$$

giving the velocity of propagation of waves of length λ in ice of thickness e, resting on water of uniform depth h.

It is remarkable that ice was the first substance for which an experimental determination of E was attempted, as described in Young's Lectures on Natural Philosophy.

8. *Waves in Water of Uniform Depth Established and Maintained by Impinging Waves of Sound in Air.*

Suppose the preceding kind of wave motion in conjunction with plane waves of sound impinging at an angle β, we have thus an illustration of a *forced*, or rather *controlled*, wave motion in the water due to *free* waves in the air. Let $\xi = B \sin\{m(x \sin\beta - z \cos\beta) - nt + \alpha\}$

represent the normal displacement in the incident wave of sound, and

$$\xi_1 = B_1 \sin\{m(x \sin\beta + z \cos\beta) - nt + \alpha_1\},$$

in the reflected wave; and let

$$\eta = b \sin(mx \sin\beta - nt)$$

represent the displacement of the surface of the water. Then, at this surface, we must have $\eta = (\xi_1 - \xi) \cos\beta,$

when $z = 0$, in order that there should be no separation of the air from the water, so that

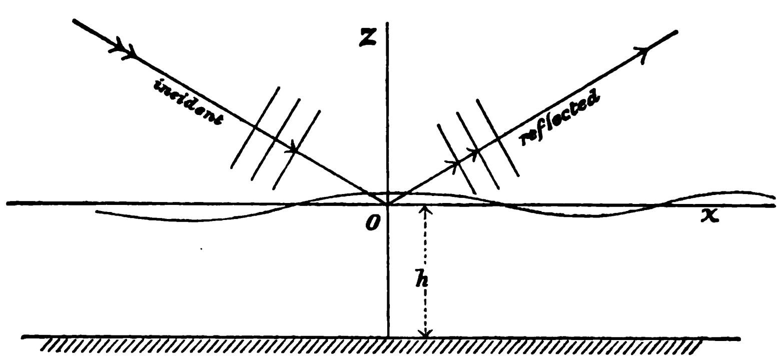
$$b \sec\beta \sin(mx \sin\beta - nt) = B_1 \sin(mx \sin\beta - nt + \alpha_1) - B \sin(mx \sin\beta - nt + \alpha)$$

for all values of x, leading to the equations

$$\left.\begin{array}{l} \beta_1 \cos\alpha_1 - B \cos\alpha = b \sec\beta \\ \beta_1 \sin\alpha_1 - B \sin\alpha = \qquad 0 \end{array}\right\}.$$

The velocity function of the motion in the water must be of the form

$$\phi = A \cosh\{m(z + h) \sin\beta\} \cos(mx \sin\beta - nt);$$



and then, since $\dfrac{d\eta}{dt} = \dfrac{\partial\phi}{\partial z},$

when $z = 0$, therefore,

$$-nb = mA \sin\beta \sinh(mh \sin\beta).$$

From the dynamical equation

$$\frac{p}{\rho} + gz + \frac{\partial\phi}{\partial t} = H$$

we have, denoting by ∂p the periodic part of p just below the surface of the water,

$$\frac{\partial p}{\rho} + g\eta + \frac{\partial\phi}{\partial t} = 0;$$

or,

$$\partial p = -g\rho\eta - \rho\frac{\partial\phi}{\partial t}$$

$$= -\{g\rho b + \rho nA \cosh(mh \sin\beta)\} \sin(mx \sin\beta - nt).$$

Suppose, now, at the surface of separation of the air and the water there is a film or cloth of tension T and superficial density σ; and we might also, if we like, suppose the film to possess flexural rigidity L like ice, without much additional complication; then, at the surface of separation,

$$\sigma\frac{d^2\eta}{dt^2} = T\frac{d^2\eta}{dx^2} - L\frac{d^4\eta}{dx^4} + \partial p - \partial p',$$

when $z = 0$, where $\partial p'$ denotes the periodic part of the pressure in the air due to the wave motion.

Now, ρ' denoting the density of air and a the velocity of the sound waves, we have

$$\frac{n^2}{m^2} = a^2 = \gamma\frac{p'}{\rho'},$$

and the cubical elasticity

$$\rho'\frac{dp'}{d\rho'} = \gamma p';$$

so that

$$\partial p' = \gamma p'\frac{\partial\rho'}{\rho'} = -\gamma p' s = -a^2\rho' s,$$

s denoting the cubical expansion; so that

$$s = \frac{\partial}{\partial x}(\xi_1 + \xi/\sin\beta) + \frac{\partial}{\partial z}(\xi_1 - \xi)\cos\beta$$

$$= mB_1 \cos\{m(x\sin\beta + z\cos\beta) - nt + \alpha_1\}$$

$$+ mB\cos\{m(x\sin\beta - z\cos\beta) - nt + \alpha\}.$$

Therefore, when $z = 0$,

$$\partial p' = -a^2\rho' m\{B_1\cos(mx\sin\beta - nt + \alpha_1) + B\cos(mx\sin\beta - nt + \alpha)\}$$

$$= a^2\rho' m(B_1\sin\alpha_1 + B\sin\alpha)\sin(mx\sin\beta - nt),$$

provided that

$$B_1\cos\alpha_1 + B\cos\alpha = 0,$$

which, combined with the previous equations, gives

$$\frac{B_1}{B} = \frac{\sin\alpha}{\sin\alpha_1} = -\frac{\cos\alpha}{\cos\alpha_1};$$

or, $$\tan\alpha = -\tan\alpha_1,\ \alpha = -\alpha_1,$$

and $$B_1 = -B = \frac{1}{2} b \sec\alpha \sec\beta.$$

Then, since $$b = -\frac{m}{n} A \sin\beta \sinh(mh\sin\beta),$$

the boundary condition becomes, when the common factor $A\sin(mx\sin\beta - nt)$ is dropped,

$$\sigma mn \sin\beta \sinh(mh\sin\beta) = T\frac{m^3}{n}\sin^3\beta \sinh(mh\sin\beta)$$
$$-L\frac{m^5}{n}\sin^5\beta\sinh(mh\sin\beta) + g\rho\frac{m}{n}\sin\beta\sinh(mh\sin\beta)$$
$$-\rho n\cosh(mh\sin\beta) - a^2\rho'\frac{m^2}{n}\tan\alpha\tan\beta\sinh(mh\sin\beta);$$

or, $$\rho' n\tan\alpha\tan\beta = -\sigma mn\sin\beta + T\frac{m^3}{n}\sin^3\beta$$
$$-L\frac{m^5}{n}\sin^5\beta + g\rho\frac{m}{n}\sin\beta - \rho n\coth(mh\sin\beta),$$

giving $\tan\alpha$, and therefore α, the change of phase of the sound wave in being reflected at the surface of the water.

9. *Reflection and Refraction of Plane Waves of Sound by a Plane Curtain.*

Suppose, now, that the cloth, instead of resting on the surface of a liquid, is the plane surface of separation of two elastic fluids of different densities ρ and ρ', but necessarily of the same pressure p when at rest, the cloth being now supposed vertical to abstract the curving effect of gravity; let us now investigate the reflection and refraction of plane waves of sound, impinging and being reflected at an angle β in the first medium of density ρ, and being refracted at an angle β' in the second medium of density ρ'.

Denoting as before by

$$\xi = B\sin\{m(x\sin\beta - z\cos\beta) - nt + \alpha\}$$

the normal displacement of the incident waves, and by

$$\xi_1 = B_1\sin\{m(x\sin\beta + z\cos\beta) - nt + \alpha_1\}$$

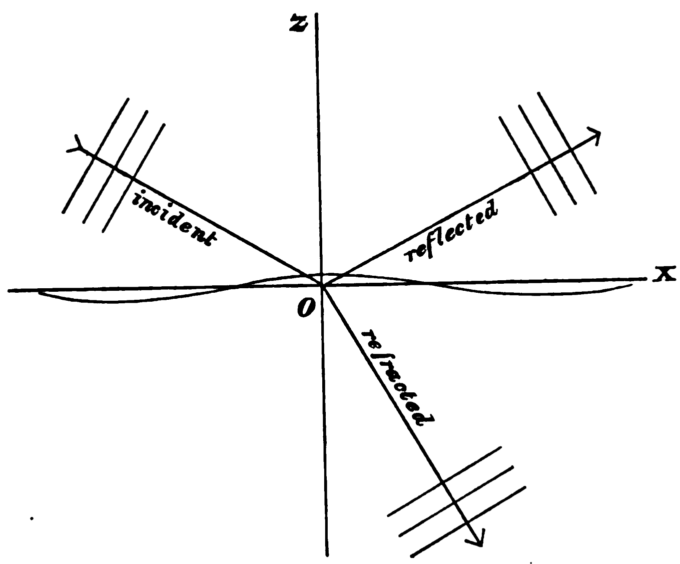
of the reflected, and by

$$\xi' = B'\sin\{m'(x\sin\beta' - z\cos\beta') - nt + \alpha'\}$$

of the refracted wave; then

(I) $$m\sin\beta = m'\sin\beta',$$ the *Law of Refraction.*

Also, denoting the displacement of the surface of separation, the cloth or curtain, by
$$\eta = b \sin (mx \sin \beta - nt);$$
then, as before, when $z = 0$,
$$\eta = (\xi_1 - \xi) \cos \beta = - \xi' \cos \beta',$$



leading to
$$\begin{aligned} B_1 \cos \alpha_1 - B \cos \alpha &= b \sec \beta, \\ B_1 \sin \alpha_1 - B \sin \alpha &= 0, \\ B' \cos \alpha' &= - b \sec \beta', \\ B' \sin \alpha' &= 0; \end{aligned}$$
so that
$$\alpha' = 0, \text{ and } B' = - b \sec \beta'.$$

For the motion of the curtain,
$$\sigma \frac{d^2\eta}{dt^2} = T \frac{d^2\eta}{dx^2} + \partial p' - \partial p,$$
where
$$\partial p = - a^2 \rho s, \quad \partial p' = - a'^2 \rho' s',$$
s and s' denoting the cubical expansion in the two media in the neighborhood of $z = 0$; also, a and a' denoting the velocity of sound in the two media. Then
$$s = m B_1 \cos (mx \sin \beta - nt + \alpha_1) + m B \cos (mx - nt + \alpha),$$
$$s' = m' B' \cos (mx \sin \beta - nt);$$
so that
$$- \sigma \frac{d^2\eta}{dt^2} + T \frac{d^2\eta}{dx^2} = (\sigma n^2 - T m^2 \sin^2 \beta) \, b \sin (mx \sin \beta - nt)$$
$$= \partial p - \partial p' = - a^2 \rho m \{ B_1 \cos (mx \sin \beta - nt + \alpha_1) + B \cos (mx - nt + \alpha) \}$$
$$+ a'^2 \rho' m' B' \cos (mx \sin \beta - nt),$$

for all values of x and t, leading to the conditions

$$(\sigma n^2 - Tm^2 \sin^2\beta)\, b = a^2\rho m\,(B_1 \sin\alpha_1 + B \sin\alpha)$$

and $$0 = -a^2\rho m\,(B_1 \cos\alpha_1 + B\cos\alpha) + a'^2\rho' m' B'.$$

Therefore, since $a^2 = n^2/m^2$, $a'^2 = n^2/m'^2$,

$$B_1 \sin\alpha_1 = B\sin\alpha = \frac{\sigma n^2 - Tm^2\sin^2\beta}{2\rho n^2}\, mb = \frac{\sigma n^2 - Tm^2\sin^2\beta}{2a^2\rho m}\, b,$$

and $$B_1\cos\alpha_1 + B\cos\alpha = \frac{a'^2\rho' m'}{a^2\rho m}\, B',$$

$$= -\frac{a'^2\rho' m'}{a^2\rho m}\, b\sec\beta';$$

also, $B_1\cos\alpha_1 - B\cos\alpha = b\sec\beta$;

so that $$B_1\cos\alpha_1 = \frac{a^2\rho m\sec\beta - a'^2\rho' m'\sec\beta'}{2a^2\rho m}\, b,$$

$$B\cos\alpha = -\frac{a^2\rho m\sec\beta + a'^2\rho' m'\sec\beta'}{2a^2\rho m}\, b,$$

giving $$\cot\alpha = -\frac{a^2\rho m\sec\beta + a'^2\rho' m'\sec\beta'}{\sigma n^2 - Tm^2\sin^2\beta},$$

$$\cot\alpha_1 = \frac{a^2\rho m\sec\beta - a'^2\rho' m'\sec\beta'}{\sigma n^2 - Tm^2\sin^2\beta},$$

whence the change of phase by reflection is determined; also,

$$B^2 : B_1^2 : B'^2$$

$$= (\sigma n^2 - Tm^2\sin^2\beta)^2 + (a^2\rho m\sec\beta + a'^2\rho' m'\sec\beta')^2,$$

$$:(\sigma n^2 - Tm^2\sin^2\beta) + (a^2\rho m\sec\beta - a'^2\rho' m'\sec\beta')^2,$$

$$: \quad 4a^4\rho^2 m^2\sec^2\beta',$$

giving the ratios of the intensities of the incident, reflected and refracted waves.

Put $\sigma = 0$ and $T = 0$ and we obtain the results of the cases considered by Green in his paper on the Reflection and Refraction of Sound, published in the Transactions of the Cambridge Philosophical Society, 1838, and republished by Ferrers in the Mathematical Papers of the late George Green, 1871.

Then $$\alpha = 0,\ \alpha_1 = 0,$$

and $$B_1 = \frac{a\rho\sec\beta - a'\rho'\sec\beta'}{2a\rho}\, b,$$

$$B = -\frac{a\rho\sec\beta + a'\rho'\sec\beta'}{2a\rho}\, b,$$

since $$am = a'm' = n.$$

We might, as in Green's paper, have supposed the incident, reflected and refracted plane waves given by

$$\phi = f\{m(x \sin \beta - z \cos \beta) - nt + \alpha\},$$
$$\phi_1 = F\{m(x \sin \beta + z \cos \beta) - nt + \alpha_1\},$$
$$\phi' = f_1\{m'(x \sin \beta' - z \cos \beta') - nt + \alpha'\},$$

and the displacement of the curtain by

$$\eta = f_2\{mx \sin \beta - nt\},$$

when f, F, f_1, f_2 denote arbitrary functions, and determine the conditions to be satisfied as before.

10. *Theory of Long Waves in Canals.*

In this theory the vertical motion of the liquid particles is supposed insensible compared with the horizontal motion, and the depth of water small compared with the wave length; so that

$$U^2 = gh,$$

as before in § 4, for water of uniform depth.

This is proved independently by supposing the pressure at any depth the same as the hydrostatic pressure due to the depth below the free surface; so that ξ, denoting the horizontal displacement of a liquid particle, and η the elevation of the free surface, then

$$\rho \frac{d^2\xi}{dt^2} = -\frac{dp}{dx}$$

and

$$\frac{dp}{dx} = g\rho \frac{d\eta}{dx};$$

so that

$$\frac{d^2\xi}{dt^2} = -g \frac{d\eta}{dx}.$$

But the equation of continuity leads to the condition

$$b(h+\eta)\left(1 + \frac{d\xi}{dx}\right) = bh;$$

or,

$$\eta + h \frac{d\xi}{dx} = 0,$$

to one order of approximation, b denoting the breadth and h the depth of the canal of water. Then

$$\frac{d^2\xi}{dt^2} = gh \frac{d^2\xi}{dx^2};$$

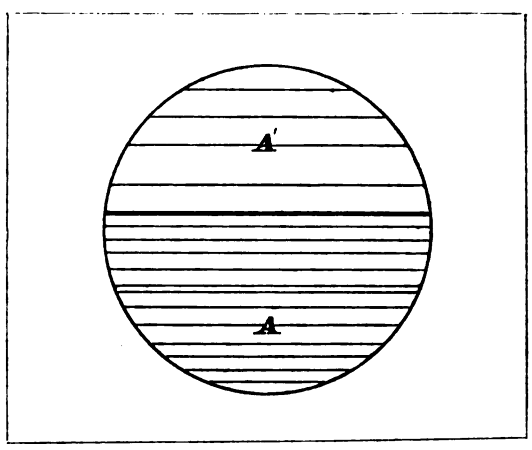
so that

$$U = gh,$$

U denoting the velocity of wave propagation.

We may, however, generalize this kind of motion, as was done by Kelland, by supposing the cross-section of the canal of any form, but uniform; and then Kelland found
$$U^2 = gA/b,$$
where A is the area of the cross-section, and b, as before, the breadth at the surface of the water.

For, in the more general case, where we consider the waves at the surface of separation of two liquids of densities ρ and ρ' filling a closed uniform horizontal pipe or conduit, so that A and A' denote the cross-sections of the pipe occupied by the liquids, and b the breadth of the plane of separation, we shall find
$$U^2 = \frac{g}{b}\frac{\rho - \rho'}{\frac{\rho}{A} + \frac{\rho'}{A'}},$$



reducing to gA/b, where $\rho' = 0$.

The simplest way to prove this is to suppose the motion made steady by applying the reversed velocity $-U$, equivalent to considering the motion relative to an origin moving with velocity U.

Then η, denoting the elevation of the surface, and u, u' the small additional velocities in the liquid due to the wave motion,
$$(A + b\eta)(U + u) = AU,$$
$$(A' - b\eta)(U + u') = A'U;$$
or,
$$Au + bU\eta = 0,$$
$$A'u' - bU\eta = 0.$$

Then, ∂p and $\partial p'$, denoting the increments of pressure caused by the wave motion in the liquids just below and just above the surface of separation,

$$\partial p + g\rho\eta + \frac{1}{2}\rho(U+u)^2 - \frac{1}{2}\rho U^2 = 0,$$

$$\partial p' + g\rho'\eta' + \frac{1}{2}\rho'(U+u')^2 - \frac{1}{2}\rho' U^2 = 0;$$

or,

$$\partial p + g\rho\eta + \rho U u = 0,$$

$$\partial p' + g\rho'\eta + \rho' U u' = 0.$$

When there is no capillarity, etc., $\partial p = \partial p'$, so that

$$g(\rho - \rho')\eta = (\rho' u' - \rho u)\, U$$

$$= \left(\frac{\rho'}{A'} + \frac{\rho}{A}\right) b U^2 \eta;$$

or,

$$U^2 = \frac{g}{b}\,\frac{\rho - \rho'}{\frac{\rho}{A} + \frac{\rho'}{A'}}.$$

But, with a separating film of tension T,

$$T\frac{d^2\eta}{dx^2} + \partial p - \partial p' = 0;$$

so, if we assume that $\eta = a \cos mx$,

we have

$$T\frac{d^2\eta}{dx^2} = -Tm^2\eta$$

$$= \partial p' - \partial p = g(\rho - \rho')\eta + (\rho u - \rho' u')\, U$$

$$= g(\rho - \rho')\eta - \left(\frac{\rho}{A} + \frac{\rho'}{A'}\right) b U^2 \eta;$$

or,

$$Tm^2 = \left(\frac{\rho}{A} + \frac{\rho'}{A'}\right) b U^2 - g(\rho + \rho').$$

When the liquids are bounded below and above by horizontal planes, at distances h and h' from the mean plane of separation, this equation becomes

$$Tm^2 = \left(\frac{\rho}{h} + \frac{\rho'}{h'}\right) U^2 - g(\rho - \rho'),$$

an equation which will be found useful as a preliminary to the consideration of the Instability of Jets and its application to the flapping of sails and flags, investigated by Lord Rayleigh (Proceedings of the London Mathematical Society, Vol. X, No. 141).

If the upper liquid had been moving with mean velocity U' different to U, the preceding equations would be replaced by

$$Au + bU\eta = 0,$$

$$A'u' - bU'\eta = 0,$$

$$\partial p + g\rho\eta + \rho U u = 0,$$

$$\partial p' + g\rho\eta + \rho' U' u' = 0,$$

and then

$$Tm^2 = \left(\rho\frac{U^2}{A} + \rho'\frac{U'^2}{A'}\right) b - g(\rho - \rho').$$

11. *Waves at the Surface of Separation of Two Liquids.*

The preceding case suggests the consideration of the general case of waves at the surface of separation of two liquids of different densities, and consequently a horizontal plane, when the liquids are either still except for the wave motion, or are flowing across each other with given mean uniform velocities, in which case the liquids must be bounded above and below by horizontal plane barriers if these velocities are not in the same direction.

First, when the liquids are still, we must have

$$\phi = A \cosh m(z + h) \cos(mx - nt)$$

in the lower liquid, of depth h, as before, and

$$\phi' = A' \cosh m(z - h') \cos(mx - nt)$$

in the upper liquid, of depth h', suppose; and then, if

$$\eta = a \sin(mx - nt)$$

represents the displacement of the surface of separation, we must have

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = \frac{\partial\phi'}{\partial z},$$

when $z = 0$, in order that there should be no separation of the liquids; consequently,

$$-na = mA \sinh mh = -mA' \sinh mh'.$$

Again, from the hydrodynamical equation

$$\frac{p}{\rho} + gz + \frac{d\phi}{dt} = H$$

we obtain, at the surface of separation,

$$\partial p + g\rho\eta + \rho\frac{d\phi}{dt} = 0$$

in the lower liquid, just below the surface of separation, and

$$\partial p' + g\rho'\eta + \rho'\frac{d\phi'}{dt} = 0$$

in the upper liquid, just above.

Neglecting capillarity, etc., $\partial p = \partial p'$, and, therefore,

$$g\rho\eta + \rho\frac{d\phi}{dt} = g\rho'\eta + \rho'\frac{d\phi'}{dt};$$

or,

$$\begin{aligned} g(\rho - \rho')\eta &= -\rho\frac{d\phi}{dt} + \rho'\frac{d\phi'}{dt} \\ &= (-\rho nA \cosh mh + \rho' nA' \cosh mh') \cos(mx - nt) \\ &= \left(\rho\frac{n^2}{m}\coth mh + \rho'\frac{n^2}{m}\coth mh'\right)\eta; \end{aligned}$$

so that

$$U^2 = \frac{n^2}{m^2} = \frac{g}{m}\,\frac{\rho - \rho'}{\rho\coth mh + \rho'\coth mh'}.$$

(Stokes, On the Theory of Oscillatory Waves, Cam. Phil. Trans., Vol. VIII, p. 441; republished in Mathematical and Physical Papers, Vol. I, p. 212.)

Putting $\rho' = 0$, we obtain

$$U^2 = \frac{g}{m} \tanh mh = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}$$

as before.

When λ is small compared with h and h', then mh and mh' are large, and we may replace coth mh and coth mh' by unity, and then

$$U^2 = \frac{g}{m} \frac{\rho - \rho'}{\rho + \rho'},$$

which, when the ratio ρ'/ρ is small, as is the case of air on water, can be replaced by

$$U^2 = \frac{g}{m}\left(1 - 2\frac{\rho'}{\rho}\right).$$

12. Suppose, now, the upper liquid is moving, like the wind, over the surface of the lower liquid with velocity V', and we wish to determine U, the velocity of propagation of waves of length λ at the surface of separation.

For generality, we shall suppose the lower liquid also moving with velocity V, and seek to determine the new relation connecting V, V' and U.

The simplest way is to take a moving origin or plane of yz, moving with velocity V in the direction of the axis of x, the direction of wave propagation, and to consider the relative motion of the liquid, which will now be steady relatively to the moving co-ordinate axes.

This is equivalent to supposing the motion made steady by impressing the reversed velocity $-U$ on the system. Then we must put

$$\phi = (V - U)x + A \cosh m(z + h) \cos mx,$$
$$\phi' = (V' - U)x + A' \cosh m(z - h') \cos mx;$$

so that, now introducing the conjugate current functions ψ and ψ',

$$\psi = (V - U)z - A \sinh m(z - h) \sin mx,$$
$$\psi' = (V' - U)z - A' \sinh m(z - h') \sin mx.$$

For the liquids not to separate, we must have, when $z = 0$, $\psi = \psi'$; so that

$$A \sinh mh = -A' \sinh mh',$$

the condition obtained otherwise by putting, when $z = 0$,

$$\frac{\partial \phi}{\partial z} = \frac{\partial \psi'}{\partial z}.$$

Now, supposing that the displacement of the surface of separation is given by

$$\eta = a \sin mx,$$

we must have, when $z = 0$,

$$\psi = (V - U)(\eta - a \sin mx),$$
$$\psi' = (V' - U)(\eta - a \sin mx);$$

so that

$$A \sinh mh = (V - U)\, a,$$
$$A' \sinh mh' = -(V' - U)\, a.$$

Also, from the hydrodynamical equations, with the same notation as before,

$$\partial p + g\rho\eta + \frac{1}{2}\rho\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\rho(V - U)^2 = 0,$$

$$\partial p' + g\rho'\eta + \frac{1}{2}\rho'\left(\frac{\partial\phi'}{\partial x}\right)^2 - \frac{1}{2}\rho'(V' - U)^2 = 0;$$

or,

$$\partial p + g\rho\eta - \rho m A (V - U) \cosh mh \sin mx = 0,$$
$$\partial p' + g\rho'\eta - \rho' m A' (V' - U) \cosh mh' \sin mx = 0;$$

or,

$$\partial p = -g\rho\eta + \rho m (V - U)^2 \coth mh\, \eta,$$
$$\partial p' = -g\rho'\eta - \rho' m (V' - U)^2 \coth mh'\, \eta.$$

If there is no capillarity, etc., at the surface of separation, $\partial p = \partial p'$; so that

$$g(\rho - \rho') = \rho m (V - U)^2 \coth mh + \rho' m (V' - U)^2 \coth mh',$$

whence U is determined, when V, V', ρ, ρ' and h, h' are given.

We have supposed here that the current velocities V and V' are in the same direction as U, the wave velocity; but if V and V' make angles α and α' with U, then, in the above expression, V and V' must be replaced by $V \cos \alpha$ and $V' \cos \alpha'$, the components $V \sin \alpha$ and $V' \sin \alpha'$ of the currents perpendicular to the direction of propagation of the waves having no effect upon the determination of U (Encyclopædia Britannica, 9th edition, article Hydromechanics).

13. In the most general case, where the surface of separation is endowed with tension T, superficial density σ and flexural rigidity L, the condition to be satisfied at this surface is

$$\sigma \frac{d^2\eta}{dt^2} = T\frac{d^2\eta}{dx^2} - L\frac{d^4\eta}{dx^4} + \partial p - \partial p'.$$

Now, if

$$\eta = a \sin mx,$$
$$\frac{d^2\eta}{dt^2} = V^2\frac{d^2\eta}{dx^2} = -n^2\eta,$$
$$\frac{d^2\eta}{dx^2} = -m^2\eta,\quad \frac{d^4\eta}{dx^4} = m^4\eta;$$

also,

$$\partial p - \partial p' = -g(\rho - \rho')\eta + \{\rho m(V - U)^2 \coth mh + \rho' m(V' - U)^2 \coth mh'\}\eta;$$

so that, omitting the common factor η,

$$\sigma n^2 - Tm^2 - Lm^4 - g(\rho - \rho') + \rho m(V - U)^2 \coth mh + \rho' m(V' - U)^2 \coth mh' = 0.$$

The application of this equation to the discussion of the Instability of Jets, including the flapping of flags and sails, has been considered by Rayleigh, as mentioned above.

In the application to a flag we may put $\rho = \rho'$, and replace coth mh and coth mh' by unity; also, $L = 0$, and we may also suppose $T = 0$; then

$$\sigma n^2 + 2\rho m(V - U)^2 = 0,$$

indicating the instability of the motion, and showing that it cannot be represented by a periodic term of small displacement; we must therefore replace in the motion $\cos(mx - nt)$ by $\cosh(mx - nt)$, $\sinh(mx - nt)$, or

$$(P \cosh mx + Q \sinh mx) \cos nt.$$

In the above general equation put $\sigma = 0$, $L = 0$, $V = 0$, and replace coth mh and coth mh' by unity; then

$$Tm^2 + g(\rho - \rho') = \rho m U^2 + \rho' m(V' - U)^2,$$

the equation considered by Thomson for the determination of the ripples produced by wind V' over the surface of still water.

If W is the velocity of ripples of the same wave length without wind,

$$Tm^2 + g(\rho - \rho') = (\rho + \rho') m W^2;$$

or,

$$W^2 = \frac{g}{m}\frac{\rho - \rho'}{\rho + \rho'} + \frac{mT}{\rho + \rho'},$$

the minimum value of which, for different values of m, is

$$W^2 = 2\sqrt{(gT)}\frac{\sqrt{(\rho - \rho')}}{\rho + \rho'},$$

and then $m^2 = g(\rho - \rho')/T$.

But

$$\rho U^2 + \rho'(V' - U)^2 = (\rho + \rho')W^2;$$

so that

$$U = \frac{\rho' V'}{\rho + \rho'} \pm \sqrt{\left\{W^2 - \frac{\rho\rho' V'^2}{(\rho + \rho')^2}\right\}},$$

giving the velocities of the ripples with and against the wind V.

The least value of V'^2 is less than $(\rho + \rho')^2/\rho\rho'$ times the least value of W^2, and, therefore, the least value of V'^2 to produce ripples is

$$2\sqrt{(gT)}\frac{\rho + \rho'}{\rho\rho'}\sqrt{(\rho - \rho')}.$$

If the wind is blowing with velocity greater than this minimum value of V', the plane surface of the water becomes unstable, and ripples are produced (Sir W. Thomson, Phil. Mag., 1871).

14. *Waves in Water Flowing with Variable Velocity fz, Some Function of the Depth z.*

In this manner we may attempt an investigation of the standing waves seen in a sloping current of water, where the velocity varies with the depth in consequence of viscosity and the fluid friction against the bottom; the method, however, will not be very rigorous, as we must begin by assuming fluid friction to account for the varying velocities at different depths, and afterwards neglect fluid friction when we come to consider the superposed wave motion.

Supposing, however, the motion is steady, we must put the current function

$$\psi = Fz - A \sinh m(z+h) \sin mx,$$

so that the mean value of u or $\frac{\partial \psi}{\partial z}$ is $F'z$ or fz, denoting fz by $F'z$. Then, at the bottom of the water,

$$\psi = Fh, \text{ a constant,}$$

and at the surface

$$\psi = F0 + \eta f0 - A \sinh mh \sin mx;$$

so that if, at the surface,

$$\eta = a \sin mx,$$

then

$$A \sinh mh = af0.$$

At the surface

$$\partial p + g\rho\eta + \frac{1}{2}\rho\left(\frac{\partial \psi}{\partial z}\right)^2 - \frac{1}{2}\rho(f0)^2 = 0;$$

or,

$$\partial p + g\rho\eta + \rho(\eta f'0 - mA \cosh mh \sin mx) f0 = 0;$$

or,

$$\partial p + \rho\eta\{g + f0f'0 - m(f0)^2 \coth mh\} = 0;$$

so that, if $\partial p = 0$,

$$g + f0f'0 - m(f0)^2 \coth mh = 0.$$

Here $f0$ denotes the velocity of the current at the surface, and $f'0$ the vertical rate of change of the velocity at the surface.

For instance, if the current flows uniformly with velocity V,

$$g - mV^2 \coth mh = 0;$$

or,

$$V^2 = \frac{g}{m} \tanh mh,$$

as before.

For a viscous liquid, flowing over a flat bottom, $f'''z = 0$, $f''z = -\frac{gi}{\mu}$, μ denoting the viscosity and i the slope of the stream, supposed small,

$$f'z = -\frac{gi}{\mu} z + C,$$

$$fz = -\frac{1}{2}\frac{gi}{\mu} z^2 + Cz + V,$$

supposing V the current velocity at the surface (Ency. Brit., Hydraulics). Therefore,

$$g + VC - mV^2 \coth mh = 0;$$

C is generally determined from the condition that the liquid adheres to the bottom, and, therefore, $fh = 0$, giving

$$C = \frac{1}{2}\frac{gi}{\mu} h - \frac{V}{h}.$$

15. In the experimental verification of the above theory of the motion of waves at the surface of separation of two liquids, we can make the wave velocity U as small as we please by making ρ and ρ' nearly equal.

Again, in order to study experimentally the waves in water of uniform depth, the best plan to obtain uniformity of depth is to pour water on the top of mercury (Stokes, Math. and Phys. Papers, I, p. 199). But in this case the mercury forming the bottom of the water will not be fixed, but will itself be set into wave motion, and the modification thus introduced is considered by Stokes on p. 217. This is a particular case of the general conditions to be satisfied when waves are propagated at the surfaces of separation of a number of superincumbent liquids forming horizontal strata, and limited above and below by fixed horizontal planes. If the upper surface is free, the density of the highest stratum of liquid must be supposed zero.

This general theorem has been worked out by Mr. R. R. Webb, and we shall proceed to investigate his results, which were given in the Math. Tripos Examination at Cambridge in Jan., 1884, as follows:

A rectangular pipe whose faces are horizontal and vertical planes is completely filled with $n+1$ liquids; show that the velocities v of propagation of waves of length λ at the surfaces of separation of the strata are given by the equation

$$\begin{vmatrix} A_1, & -B_2, & 0, & 0, & 0, & 0 & . & . & . & . \\ -B_2, & A_2, & -B_3, & 0, & 0, & 0 & . & . & . & . \\ 0, & -B_3, & A_3, & -B_4, & 0, & 0 & . & . & . & . \\ 0, & 0, & -B_4, & A_4, & -B_5, & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0, & -B_{n-1}, & A_{n-1}, & -B_n \\ . & . & . & . & . & . & 0, & 0, & -B_n, & A_n \end{vmatrix} = 0,$$

where $A_r = 2\pi v^2/\lambda\,(\rho_{r+1}.\ \coth 2\pi h_{r+1}/\lambda + \rho_r \coth 2\pi h_r/\lambda) - g\,(\rho_{r+1} - \rho_r)$,
$B_r = 2\pi v^2 \rho_r/\lambda$ cosech $2\pi h_r/\lambda$,
and h_r is the equilibrium thickness of the r^{th} stratum of density ρ_r.

In particular, if $\rho_r = r\sigma$, $h_r = ra$, then the $2n$ values of v are included in the formula

$$v = \pm \frac{1}{2}\sqrt{(ga)}\sec\left(\frac{r}{n+1}\frac{\pi}{2}\right),$$

where r is supposed to assume the values $1, 2, 3, \ldots. n$, and λ, the wave length, is supposed very large compared with na.

In order that the velocity function ϕ_r in the r^{th} stratum should satisfy the equation of continuity and the conditions that

$\dfrac{\partial\phi_r}{\partial z} = \dfrac{\partial\phi_{r-1}}{\partial z}$ at the surface of separation of the r^{th} and $r - 1^{\text{th}}$ liquids,

$\dfrac{\partial\phi_r}{\partial z} = \dfrac{\partial\phi_{r+1}}{\partial z}$ " " " " r^{th} " $r + 1^{\text{th}}$ "

then, taking the plane of xy in the upper surface of the pipe, we must have

$$\phi_r = \{ C_{r-1}\cosh m\,(z + h_1 + h_2 + \ldots. + h_r)$$
$$- C_r \cosh m\,(z + h_1 + h_2 + \ldots. + h_{r-1})\}\,\frac{\cos(mx - nt)}{m \sinh mh_r},$$

and then

$$\frac{\partial\phi_r}{\partial z} = C_{r-1}\cos(mx - nt),\ \text{when } z = -h_1 - h_2 - \ldots. - h_{r-1},$$

$$\frac{\partial\phi_r}{\partial z} = C_r \cos(mx - nt),\ \text{when } z = -h_1 - h_2 - \ldots. - h_r;$$

and, therefore, if η_r denotes the elevation of the surface of separation between the r^{th} and $r + 1^{\text{th}}$ liquids,

$$\frac{d\eta_r}{dt} = C_r \cos(mx - nt),$$

and

$$\eta_r = -\frac{C_r}{n}\sin(mx - nt).$$

To express the fact that there is no discontinuity at this surface of separation, we have the equations

$$\frac{\partial p_r}{\rho_r} + g\eta_r + \frac{\partial\phi_r}{\partial t} = 0,$$

$$\frac{\partial p_{r+1}}{\rho_{r+1}} + g\eta_r + \frac{\partial\phi_{r+1}}{\partial t} = 0;$$

so that, since $\partial p_r = \partial p_{r+1}$,

$$g\rho_r\eta_r + \rho_r\frac{\partial\phi_r}{\partial t} = g\rho_{r+1}\eta_r + \rho_{r+1}\frac{\partial\phi_{r+1}}{\partial t};$$

or,

$$g\,(\rho_{r+1} - \rho_r)\,\eta_r + \rho_{r+1}\frac{\partial\phi_{r+1}}{\partial t} - \rho_r\frac{\partial\phi_r}{\partial t} = 0.$$

Now, when $z = -h_1 - h_2 - \ldots - h_r$,

$$\frac{\partial\phi_r}{\partial t} = \frac{n}{m}(C_{r-1}\operatorname{cosech} mh_r - C_r \coth mh_r)\sin(mx - nt),$$

$$\frac{\partial\phi_{r+1}}{\partial t} = \frac{n}{m}(C_r \coth mh_{r+1} - C_{r+1}\operatorname{cosech} mh_{r+1})\sin(mx - nt);$$

so that, dropping the common factor $\frac{1}{n}\sin(mx - nt)$,

$$-g(\rho_{r+1} - \rho_r)C_r + \frac{n^2}{m}(C_r\rho_{r+1}\coth mh_{r+1} - C_{r+1}\rho_{r+1}\operatorname{cosech} mh_{r+1}$$
$$- C_{r-1}\rho_r \operatorname{cosech} mh_r + C_r\rho_r \coth mh_r) = 0;$$

or,
$$-C_{r-1}B_r + C_rA_r - C_{r+1}B_{r+1} = 0,$$

since
$$n^2/m = 2\pi v^2/\lambda, \quad m = 2\pi/\lambda.$$

Also, the top and bottom of the pipe being fixed horizontal planes, $C_0 = 0$ and $C_{n+1} = 0$, so that the elimination of the C's leads to the determinant given above.

When the pipe is open at the top, we can represent the motion by supposing $\rho_1 = 0$, or $B_1 = 0$, and then the particular case for waves when there are two superincumbent liquids with a free surface has been given by Stokes (Math. and Phys. Papers, I, p. 217).

It will be noticed that, although there is no discontinuity in the value of $\frac{\partial\phi}{\partial z}$ at a surface of separation of two strata, there is discontinuity in $\frac{\partial\phi}{\partial x}$, denoting a slipping of one surface over the other, the slipping velocity at the r^{th} surface, where $z = -h_1 - h_2 - \ldots - h_r$, being

$$\frac{\partial\phi_{r+1}}{\partial x} - \frac{\partial\phi_r}{\partial x}$$
$$= \{C_{r-1}\operatorname{cosech} mh_r - C_r(\coth mh_{r+1} + \coth mh_r) + C_{r+1}\operatorname{cosech} mh_{r+1}\}\sin(mx - nt).$$

This slipping proves a difficulty in the attempt of proceeding from the above investigation of waves in strata of finite thickness to the case of waves in a liquid of variable density arranged in horizontal strata.

When λ is large and m is consequently small, we may replace $\rho_r \coth mh_r$ and $\rho_r \operatorname{cosech} mh_r$ by ρ_r/mh_r; and then, if $\rho_r = r\sigma$, $h_r = ra$,

$$A_r = 2\sigma\frac{n^2}{m^2} - g\sigma a = 2\sigma v^2 - g\sigma a,$$

$$B_r = \sigma\frac{n^2}{m^2} = \sigma v^2;$$

so that the above determinant becomes

$$\begin{vmatrix} C, & 1, & 0, & 0, & 0 & . & . \\ 1, & C, & 1, & 0, & 0 & . & . \\ 0, & 1, & C, & 0, & 0 & . & . \\ 0, & 0, & 1, & C, & 1 & . & . \\ . & . & . & . & . & . & . \end{vmatrix} n \text{ rows}$$

$$= 0, \text{ where } C = 2 - \frac{ga}{v^2},$$

the determinant considered by Rayleigh (Theory of Sound, I, p. 131).

It is there proved that, putting $C = 2 \cos \vartheta$, so that

$$v^2 = \frac{1}{4} ga \sec^2 \frac{1}{2} \vartheta,$$

then
$$\vartheta = \frac{r\pi}{n+1},$$
where
$$r = 1, 2, 3, \ldots . n.$$

16. *Waves in Canals with Sloping Sides, or against a Sloping Beach.*

So far, the wave motion considered has only involved two co-ordinates, x and z, and might be considered limited by any two fixed vertical planes perpendicular to the axis of y.

In the case of the canal of uniform arbitrary cross-section, Kelland obtained the equation,
$$U^2 = gA/b,$$
for U, the wave velocity of long waves moving along the canal.

Kelland, however, was successful in obtaining an exact expression for the motion of progressive waves in a straight canal the sides of which sloped down uniformly to an edge at an angle of 45°; he found that, taking the axis of x along this edge, we can put

$$\phi = A \cosh my \cosh mz \cos \sqrt{2}\,(mx - nt),$$

satisfying the equation of continuity

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0,$$

and also the boundary condition $\dfrac{\partial\phi}{\partial\nu} = 0$;

or,
$$l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z} = 0,$$

l, m, n denoting the direction cosines of the normal to the boundary. In this case the boundary conditions are

$$\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial z} = 0, \text{ when } y - z = 0,$$

$$\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z} = 0, \text{ when } y + z = 0,$$

which are immediately seen to be satisfied.

At the surface of the water, where $z = h$,

$$l\frac{\partial\phi}{\partial z} = \phi,$$

for all values of x and y; or,

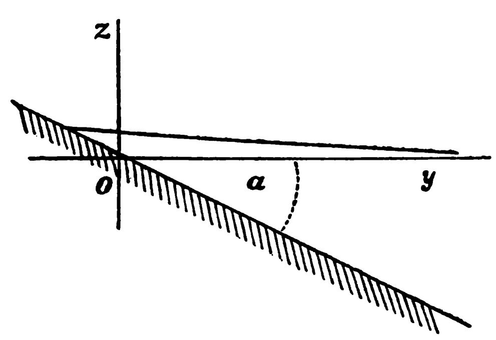
$$ml \sinh mh = \cosh mh, \text{ or } ml = \coth mh,$$

and
$$U^2 = \frac{n^2}{m^2} = \frac{g}{lm^2} = \frac{g}{m}\tanh mh,$$

the same as for waves in water of depth h, but now $2\pi/\lambda = m\sqrt{2}$.

By transferring the axis of x to the edge of the water on one bank, we obtain
$$\phi = A \cosh m(y + h) \cosh m(z - h) \cos \sqrt{2}(mx - nt),$$
which, when h is made indefinitely great, can be replaced by
$$\phi = Be^{-m(y-z)} \cos \sqrt{2}(mx - nt),$$
giving the motion for waves moving parallel to a shore sloping at 45°, the crests of the waves being perpendicular to the shore.

This may easily be generalized, as Stokes has shown (Report on Recent Researches in Hydrodynamics, Math. and Phys. Papers, I, p. 167), for a shore sloping at any angle α, by putting
$$\phi = Be^{-m(y\cos\alpha - z\sin\alpha)} \cos(mx - nt),$$



satisfying the equation of continuity and the boundary condition

$$\frac{\partial\phi}{\partial y}\sin\alpha + \frac{\partial\phi}{\partial z}\cos\alpha = 0, \text{ when } y\sin\alpha + z\cos\alpha = 0;$$

also, at the free surface $z = 0$,

$$l\frac{\partial\phi}{\partial z} = \phi;\ \text{or,}\ ml \sin\alpha = 1;$$

so that $$U^2 = \frac{n^2}{m^2} = \frac{g}{lm^2} = \frac{g}{m}\sin\alpha = \frac{g\lambda}{2\pi}\sin\alpha,$$

α denoting here the slope of the shore to the horizon.

Analogous to Kelland's previous solution, we might put

$$\phi = A \sinh my \sinh mz \sin\sqrt{2}\,(mx - nt),$$

satisfying the equation of continuity and the boundary conditions, and at the free surface $z = h$, $l\frac{\partial\phi}{\partial z} = \phi$ gives

$$ml = \tanh mh;$$

or, $$U^2 = \frac{g}{m}\coth mh.$$

The shape of the free surface will be different in the two cases; in the first, Kelland's case, when $z = h$,

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = mA \sinh mh \cosh my \cos\sqrt{2}\,(mx - nt);$$

so that $$\eta = -\frac{mA}{n\sqrt{2}}\sinh mh \cosh my \sin\sqrt{2}\,(mx - nt)$$

or the form $\eta = a \cosh my \sin\sqrt{2}\,(mx - nt)$,

an even function of y; and in the second case,

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = mA \cosh mh \sinh my \sin\sqrt{2}\,(mx - nt);$$

so that the free surface is of the form

$$\eta = a \sinh my \cos\sqrt{2}\,(mx - nt),$$

an odd function of y.

Introducing capillarity on the free surface, but neglecting its effect at the contact of the surface with the bank, then the equation

$$l\frac{\partial\phi}{\partial z} + \frac{Tl}{g\rho}\frac{\partial^3\phi}{\partial z^3} = \phi$$

gives $$ml \sinh mh + \frac{Tlm^3}{g\rho}\sinh mh = \cosh mh;$$

or, $$ml + \frac{Tlm^3}{g\rho} = \coth mh;$$

and, for the second kind of motion,

$$ml + \frac{Tlm^3}{g\rho} = \tanh mh.$$

17. Let us apply Kelland's expression to determine the progressive waves at the surface of separation of two liquids, each half filling a pipe, of which the cross-section is a square with a vertical diagonal, of length $2h$.

Taking diagonals of the square as axes of y and z, then we can put

$$\phi = A \cosh my \cosh m(z+h) \cos \sqrt{2}(mx - nt),$$
$$\phi' = -A \cosh my \cosh m(z-h) \cos \sqrt{2}(mx - nt),$$

satisfying the equation of continuity and the boundary conditions, except just where the surface of separation meets the boundary, the disturbing effect of which we shall neglect, although of course disturbing waves would be generated thereby.

Then, as before, if η denotes the elevation of the surface of separation where the mean value of $z = 0$,

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = mA \cosh my \sinh mh \cos \sqrt{2}(mx - nt);$$

so that

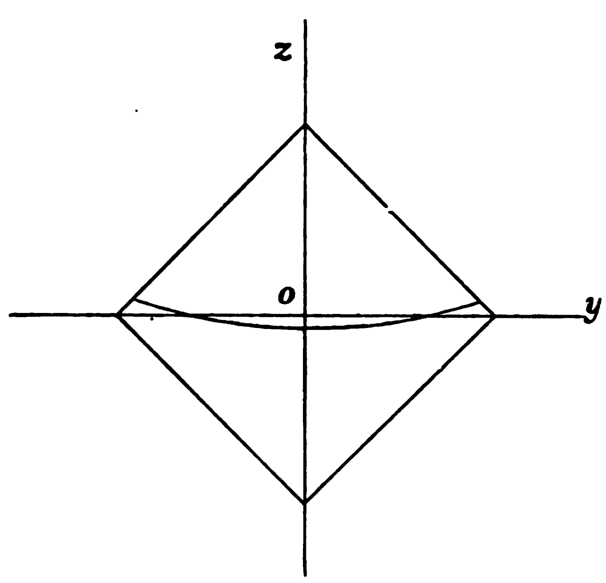
$$\eta = -\frac{mA}{n\sqrt{2}} \cosh my \sinh mh \sin \sqrt{2}(mx - nt);$$

also,

$$\frac{d\phi}{dt} = nA\sqrt{2} \cosh my \cosh mh \sin \sqrt{2}(mx - nt) = -2\frac{n^2}{m}\eta \coth mh,$$

and

$$\frac{d\phi'}{dt} = 2\frac{n^2}{m}\eta \coth mh.$$



Also, as before,

$$\partial p + g\rho\eta + \rho\frac{d\phi}{dt} = 0,$$

$$\partial p' + g\rho'\eta + \rho'\frac{d\phi'}{dt} = 0;$$

or,

$$\partial p + g\rho\eta - 2\rho\frac{n^2}{m}\eta\coth mh = 0,$$

$$\partial p' + g\rho'\eta + 2\rho'\frac{n^2}{m}\eta\coth mh = 0;$$

so that if $\partial p = \partial p'$,

$$g(\rho - \rho') = 2\frac{n^2}{m}(\rho + \rho')\coth mh;$$

or,

$$U^2 = \frac{n^2}{m^2} = \frac{1}{2}\frac{g}{m}\frac{\rho - \rho'}{\rho + \rho'}\tanh mh.$$

Similarly, by putting

$$\phi = A\sinh my\sinh m(z+h)\sin\sqrt{2}(mx - nt),$$
$$\phi' = A\sinh my\sinh m(z-h)\sin\sqrt{2}(mx - nt),$$

we should obtain

$$U^2 = \frac{1}{2}\frac{g}{m}\frac{\rho - \rho'}{\rho + \rho'}\coth mh.$$

18. *Standing Waves across a Rectangular Channel.*

We shall find that, by replacing the hyperbolic functions of y and z partly by circular functions in the above solutions for progressive waves along the channel, we shall be able to solve the motion of standing waves in which the crests are parallel to the axis of the canal or channel. For, if we put

$$\phi = A(\cos my\cosh mz + \cosh my\cos mz)\cos nt,$$

or

$$\phi = A(\sin my\sinh mz + \sinh my\sin mz)\sin nt,$$

expressions which are equivalent to those obtained by Kirchoff (Ueber stehende Schwingungen einer schweren Flüssigkeit, Gesammelte Abhandlungen, II, p. 440), we shall satisfy the equation of continuity and the boundary conditions

$$\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial z} = 0, \text{ when } y - z = 0,$$

$$\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z} = 0, \text{ when } y + z = 0;$$

and at the free surface $z = h$ the condition

$$l\frac{\partial\phi}{\partial z} = \phi,$$

for all values of y, leads to the equations

$$ml(\cos my \sinh mh - \cosh my \sin mh) = \cos my \cosh mh + \cosh my \cos mh,$$

giving $\quad ml \sinh mh = \cosh mh$ and $-ml \sin mh = \cos mh,$

or $$ml = \coth mh = -\cot mh,$$

for the even vibrations, and

$$ml(\sin my \cosh mh + \sinh my \cos mh) = \sin my \sinh mh + \sinh my \sin mh,$$

giving $\quad ml \cosh mh = \sinh mh$ and $ml \cos mh = \sin mh,$

or $$ml = \tanh mh = \tan mh,$$

for the odd vibrations.

In these equations mh is the same as Kirchoff's p, and, with the notation of the hyperbolic functions, Kirchoff's period equations

$$\sinh p = \frac{\sin p}{\sqrt{(\cos 2p)}}$$

and $$\cosh p = \frac{\cos p}{\sqrt{(\cos 2p)}}$$

correspond to our period equations

$$\coth p = -\cot p$$

and $$\tanh p = \tan p;$$

both being included in the single equation

$$\cos 2p \cosh 2p = 1,$$

the period equation for the lateral vibrations of a free-free or clamped-clamped bar (Rayleigh, Sound, I, p. 219).

In fact, the vibrations of the surface are of exactly the same character as those of a free-free bar of length $2h$, the first value of ϕ giving the even and the second the odd vibrations.

Suppose the surface were covered with ice of thickness e and flexural rigidity $L = \frac{1}{12} e^3 E$; then, at the surface $z = h$,

$$\rho e \frac{d^2\eta}{dt^2} = -L \frac{d^4\eta}{dx^4} + \partial p$$

and $$\partial p + g\rho\eta + \rho \frac{\partial \phi}{\partial t} = 0.$$

But $$\frac{d^2\eta}{dt^2} = -n^2\eta, \quad \frac{d^4\eta}{dx^4} = m^4\eta;$$

so that $$n^2\rho e\eta = m^4 L\eta + g\rho\eta + \rho \frac{\partial \phi}{\partial t},$$

or $$\{\rho(g-n^2e)+m^4L\}\,\eta=-\rho\frac{\partial\phi}{\partial t};$$

and since $$\frac{d\eta}{dt}=\frac{\partial\phi}{\partial z},\ \frac{\partial^2\phi}{\partial t^2}=-n^2\phi,$$

$$\{\rho(g-n^2e)+m^4L\}\frac{\partial\phi}{\partial z}=n^2\rho\phi;$$

or since $g=ln^2$,

$$\left(l-e+\frac{m^4L}{n^2\rho}\right)\frac{\partial\phi}{\partial z}=\phi,$$

and, therefore, as before,

$$m(l-e)+\frac{m^5L}{n^2\rho}=\coth mh=-\cot mh,$$

or $$=\tanh mh=\tan mh,$$

showing that the length of the equivalent simple pendulum is altered by $m^4.L/n^2\rho-e$ by the presence of the ice.

Dropping for the present the factors $A\cos nt$ and $A\sin nt$, then

$$\phi=\cos my\cosh mz+\cosh my\cos mz,$$

or $$\phi=\sin my\sinh mz+\sinh my\sin mz,$$

and, therefore, the conjugate current functions are

$$\psi=\sin my\sinh mz-\sinh my\sin mz,$$

or $$\psi=-\cos my\cosh mz+\cosh my\cos mz;$$

so that $$\phi+i\psi=\cos m(z+iy)+\cosh m(z+iy),$$

or $$=i\cos m(z+iy)-i\cosh m(z+iy).$$

Denoting $\phi+i\psi$ by w, and $z+iy$ by u, then

$$w=\cos mu+\cos imu,$$

or $$w=i\cos mu-i\cos imu,$$

gives the required motion in a rectangular channel.

By transferring the axis of x to the edge of the water on one bank, we obtain $\phi=\cos m(y-h)\cosh m(z+h)+\cosh m(y-h)\cos m(z+h)$,

or $\phi=\sin m(y-h)\sinh m(z+h)+\sinh m(y-h)\sin m(z+h)$;

and these are the co-ordinates employed by Kirchoff.

When h is made indefinitely great, these expressions may be replaced by (omitting constant factors, and remembering that the axis of z is drawn vertically *upwards*),

$$\phi=e^{mz}(\cos my-\sin my)+e^{-my}(\cos mz-\sin mz),$$

and then, when $z=0$, $$l\frac{\partial\phi}{\partial z}=\phi,$$

if $$ml=1,$$

giving Kirchoff's solution of standing waves parallel to a shore sloping at 45° (Abhandlungen, II, p. 434).

Returning to the original axes, the second value of ϕ gives, when m is small, so that $m^2, \ldots$ may be neglected,

$$\phi = 2myz;$$

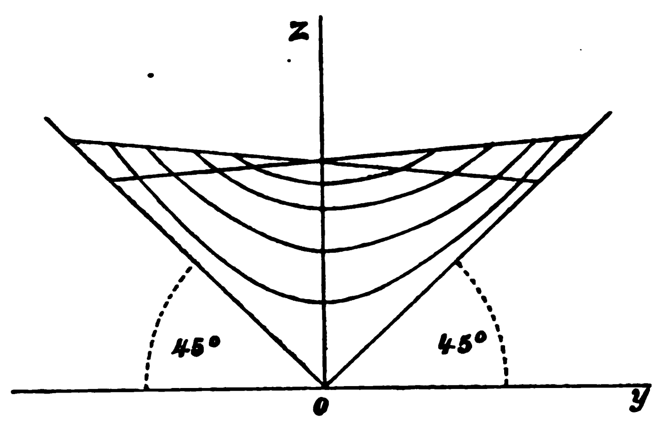
or, restoring the periodic factor, we may put

$$\phi = 2myz \sin nt,$$

and then $$l = h;$$

also, $$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = 2my \sin nt;$$

so that $$\eta = -2\frac{m}{n} y \cos nt,$$



so that the free surface of the liquid remains plane during this kind of wave motion. Also, $$\psi = m(y^2 - z^2) \sin nt,$$

so that the liquid particles oscillate in rectangular hyperbolas (Kirchoff, Abhandlungen, II, p. 436).

19. *Standing Waves across a Channel of* 120°.

Let us now attempt the solution of the corresponding waves in a canal the sides of which slope at angles of 30° to the horizon, and are therefore inclined to each other at an angle of 120°.

First, we notice that we can begin with an algebraical solution by putting, as the simplest case,

$$\phi = A\Phi \cos nt,$$

where $$\Phi = z^3 - 3y^2z + h^3,$$

and the corresponding stream function

$$\Psi = 3yz^2 - y^3$$
$$= y\,(z\sqrt{3} - y)(z\sqrt{3} + y),$$

which vanishes when $y = \pm z\sqrt{3}$, showing that the boundary conditions are satisfied, and also,

$$\Phi + i\Psi = (z + iy)^3 + h^3.$$

Then, at the free surface $z = h$,

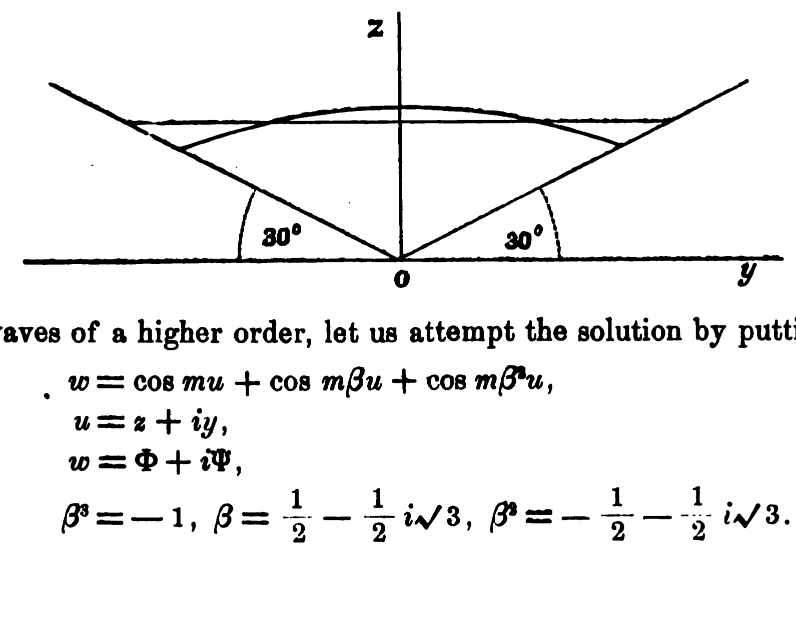
$$\frac{\partial\Phi}{\partial z} = 3h^2 - 3y^2, \quad \Phi = 2h^3 - 3hy^2;$$

so that $$\frac{\partial\Phi}{\partial z} = \Phi,$$

and, therefore, $l = h$.

The free surface is now a parabolic cylinder, for

$$\frac{d\eta}{dt} = 3A\,(h^2 - y^2) \cos nt,$$
$$\eta = \frac{3A}{n}(h^2 - y^2) \sin nt.$$



For waves of a higher order, let us attempt the solution by putting

$$w = \cos mu + \cos m\beta u + \cos m\beta^2 u,$$

where $$u = z + iy,$$
$$w = \Phi + i\Psi,$$

and $$\beta^3 = -1, \; \beta = \frac{1}{2} - \frac{1}{2}i\sqrt{3}, \; \beta^2 = -\frac{1}{2} - \frac{1}{2}i\sqrt{3}.$$

Then, at the boundary $y = z\sqrt{3}$,

$$w = \cos(1 + i\sqrt{3})\,mz + \cos\frac{1}{2}(1 - i\sqrt{3})(1 + i\sqrt{3})\,mz + \cos\frac{1}{2}(1 + i\sqrt{3})^2 mz$$

$$= \cos(1 + i\sqrt{3})\,mz + \cos 2mz + \cos(-1 + i\sqrt{3})\,z,$$

a real quantity, so that $\Psi = 0$.

Again, at the boundary $y = -z\sqrt{3}$, w is real and $\psi = 0$. These conditions will also be satisfied by putting

$$w = \sin mu - \sin m\beta u + \sin m\beta^2 u,$$

so that generally we can put

$$w = \sin m(u - \alpha) - \sin m(\beta u + \alpha) + \sin m(\beta^2 u - \alpha);$$

so that, since

$$\beta u + \alpha = \frac{1}{2}(y\sqrt{3} + z + 2\alpha) + \frac{1}{2} i (y - z\sqrt{3}),$$

$$\beta^2 u - \alpha = \frac{1}{2}(y\sqrt{3} - z - 2\alpha) - \frac{1}{2} i (y + z\sqrt{3}),$$

$$\Phi = \sin m(z - \alpha)\cosh my - \sin\frac{1}{2}m(y\sqrt{3} + z + 2\alpha)\cosh\frac{1}{2}m(y - z\sqrt{3})$$
$$+ \sin\frac{1}{2}m(y\sqrt{3} - z - 2\alpha)\cosh\frac{1}{2}m(y + z\sqrt{3}),$$

$$\Psi = \cos m(z - \alpha)\sinh my - \cos\frac{1}{2}m(y\sqrt{3} + z + 2\alpha)\sinh\frac{1}{2}m(y - z\sqrt{3})$$
$$- \cos\frac{1}{2}m(y\sqrt{3} - z - 2\alpha)\sinh\frac{1}{2}m(y + z\sqrt{3}).$$

Putting $y = z\sqrt{3}$,

$$\Psi = \cos m(z - \alpha)\sinh mz\sqrt{3} - \cos m(z - \alpha)\sinh mz\sqrt{3} = 0,$$

and putting $y = -z\sqrt{3}$,

$$\Psi = -\cos m(z - \alpha)\sinh mz\sqrt{3} - \cos m(-z + \alpha)\sinh(-mz\sqrt{3}) = 0,$$

so that the boundary conditions are satisfied.

Expanded in ascending powers of $(z + ix)$, we shall find

$$w = -3\sin\alpha - \frac{1}{2}m^3(z + ix)^3\cos\alpha + \ldots.,$$

so that when m is small, and $m^5, \ldots.$ can be neglected, we obtain the previous algebraical solution.

At the free surface $z = h$ we must have

$$l\frac{\partial\Phi}{\partial z} = \Phi$$

for all values of y; and, therefore, since we may write

$$\Phi = \sin m(z-a)\cosh my$$
$$-2\sin\frac{1}{2}m(z+2a)\cosh\frac{1}{2}mz\surd 3\cosh\frac{1}{2}my\cos\frac{1}{2}my\surd 3$$
$$+2\cos\frac{1}{2}m(z+2a)\sinh\frac{1}{2}mz\surd 3\sinh\frac{1}{2}my\sin\frac{1}{2}my\surd 3,$$

$$l\frac{\partial\Phi}{\partial z} = ml\cos m(z-a)\cosh my$$
$$-ml\left\{\cos\frac{1}{2}m(z+2a)\cosh\frac{1}{2}mz\surd 3+\surd 3\sin\frac{1}{2}m(z+2a)\right.$$
$$\left.\sinh\frac{1}{2}mz\surd 3\right\}\cosh\frac{1}{2}my\cos\frac{1}{2}my\surd 3$$
$$+ml\left\{-\sin\frac{1}{2}m(z+2a)\sinh\frac{1}{2}mz\surd 3+\surd 3\cos\frac{1}{2}m(z+2a)\right.$$
$$\left.\cosh\frac{1}{2}mz\surd 3\right\}\sinh\frac{1}{2}my\sin\frac{1}{2}mz\surd 3;$$

therefore, at the free surface $z=h$,

$$ml\cos m(h-a) = \sin m(h-a), \tag{I}$$

$$ml\left\{\cos\frac{1}{2}m(h+2a)\cosh\frac{1}{2}mh\surd 3+\surd 3\sin\frac{1}{2}m(h+2a)\sinh\frac{1}{2}mh\surd 3\right\}$$
$$=2\sin\frac{1}{2}m(h+2a)\cosh\frac{1}{2}mh\surd 3, \tag{II}$$

$$ml\left\{-\sin\frac{1}{2}m(h+2a)\sinh\frac{1}{2}mh\surd 3+\surd 3\cos\frac{1}{2}m(h+2a)\cosh\frac{1}{2}mh\surd 3\right\}$$
$$=2\cos\frac{1}{2}m(h+2a)\sinh\frac{1}{2}mh\surd 3, \tag{III}$$

three equations for determining mh, a and l.

From (II) and (III), eliminating ml,

$$\cos^2\frac{1}{2}m(h+2a)\sinh mh\surd 3+\surd 3\sin m(h+2a)\sinh^2\frac{1}{2}mh\surd 3$$
$$=-\sin^2\frac{1}{2}m(h+2a)\sinh mh\surd 3+\surd 3\sin m(h+2a)\cosh^2\frac{1}{2}mh\surd 3;$$

or,

$$\sinh mh\surd 3 = \surd 3\sin m(h+2a). \tag{IV}$$

Also, from (II) and (III),

$$\tan\frac{1}{2}m(h+2a) = \frac{ml\cosh\frac{1}{2}mh\surd 3}{2\cosh\frac{1}{2}mh\surd 3 - ml\surd 3\sinh\frac{1}{2}mh\surd 3}$$
$$=\frac{ml\surd 3\cosh\frac{1}{2}mh\surd 3 - 2\sinh\frac{1}{2}mh\surd 3}{ml\sinh\frac{1}{2}mh\surd 3};$$

so that

$$\frac{1}{2} m^2l^2 \sinh mh\sqrt{3} = 2ml\sqrt{3}\left(\cosh^2 \frac{1}{2} mh\sqrt{3} + \sinh^2 \frac{1}{2} mh\sqrt{3}\right)$$
$$- \frac{3}{2} m^2l^2 \sinh mh\sqrt{3} - 2\sinh mh\sqrt{3};$$

or,
$$ml + \frac{1}{ml} = \sqrt{3}\coth mh\sqrt{3}. \tag{V}$$

From (I), $ml = \tan m(h-a)$,

so that

$$ml + \frac{1}{ml} = 2\operatorname{cosec} 2m(h-a);$$

or,
$$\sin 2m(h-a) = \frac{2}{\sqrt{3}}\tanh mh\sqrt{3}. \tag{VI}$$

From (IV) and (VI),

$$\sin mh \cos 2ma + \cos mh \sin 2ma = \frac{1}{\sqrt{3}}\sinh mh\sqrt{3},$$
$$\sin 2mh \cos 2ma - \cos 2mh \sin 2ma = \frac{2}{\sqrt{3}}\tanh mh\sqrt{3};$$

and, therefore,

$$\sqrt{3}\sin 3mh \cos 2ma = \sinh mh\sqrt{3}\left(\cos 2mh + \frac{2\cos mh}{\cosh mh\sqrt{3}}\right),$$
$$\sqrt{3}\sin 3mh \sin 2ma = \sinh mh\sqrt{3}\left(\sin 2mh - \frac{2\sin mh}{\cosh mh\sqrt{3}}\right).$$

Squaring and adding, to eliminate a,

$$\frac{3\sin^2 3mh}{\sinh^2 mh\sqrt{3}} = 1 + \frac{4\cos 3mh}{\cosh mh\sqrt{3}} + \frac{4}{\cosh^2 mh\sqrt{3}};$$

or, denoting $\cos 3mh$ by α, and $\cosh mh\sqrt{3}$ by β,

$$3\frac{1-\alpha^2}{\beta^2-1} = 1 + 4\frac{\alpha}{\beta} + \frac{4}{\beta^2},$$

reducing to

$$\beta^4 + 4\alpha\beta^3 + 3\alpha^2\beta^2 - 4\alpha\beta - 4 = 0,$$

or
$$(\beta^2 + 2\alpha\beta)^2 = (\alpha\beta + 2)^2,$$
$$\beta^2 + 2\alpha\beta = \pm(\alpha\beta + 2);$$

so that
$$\beta^2 + \alpha\beta = 2,$$

or
$$\beta^2 + 3\alpha\beta = -2;$$
$$\alpha = -\beta + 2/\beta,$$

or
$$3\alpha = -\beta - 2/\beta;$$
$$\cos 3mh = -\cosh mh\sqrt{3} + 2\operatorname{sech} mh\sqrt{3},$$

or
$$3\cos 3mh = -\cosh mh\sqrt{3} - 2\operatorname{sech} mh\sqrt{3},$$

the period equations.

Having found a value of mh from these equations, then α and ml are determined.

On examination, however, the second of these period equations will be found to have no real roots, while the only real roots of the first equation will be found to be given by $mh = 0$, repeated six times, giving the previous algebraical solution; this is seen by investigating the intersections of the curves $y = \cot x\sqrt{3}$ and $z = -\cosh x + 2 \operatorname{sech} x$, or $y = 3 \cos x\sqrt{3}$ and $z = -\cosh x - 2 \operatorname{sech} x$, where $x = mh\sqrt{3}$.

If, however, we put

$$w = \sinh m(u - \alpha) - \sinh m(\beta u + \alpha) + \sinh m(\beta^2 u - \alpha),$$

then

$$\Phi = \sinh m(z - \alpha) \cos my - \sinh \frac{1}{2} m(y\sqrt{3} + z + 2\alpha) \cos \frac{1}{2} m(y - z\sqrt{3})$$
$$+ \sinh \frac{1}{2} m(y\sqrt{3} - z - 2\alpha) \cos \frac{1}{2} m(y + z\sqrt{3});$$

$$\Psi = -\cosh m(z - \alpha) \sin my + \cosh \frac{1}{2} m(y\sqrt{3} + z + 2\alpha) \sin \frac{1}{2} m(y - z\sqrt{3})$$
$$+ \cosh \frac{1}{2} m(y\sqrt{3} - z - 2\alpha) \sin \frac{1}{2} m(y + z\sqrt{3});$$

so that $\Psi = 0$, when $y = \pm z\sqrt{3}$, and the boundary conditions are satisfied.

Then, exactly as before, from the free surface condition

$$l\frac{\partial \Phi}{\partial z} = \Phi,$$

when $z = h$, for all values of y, we shall find

$$ml = \tanh m(h - \alpha), \tag{I}$$

$$\sin mh\sqrt{3} = \sqrt{3} \sinh m(h + 2\alpha), \tag{IV}$$

$$ml - \frac{1}{ml} = -\sqrt{3} \cot mh\sqrt{3}, \tag{V}$$

$$\sinh 2m(h - \alpha) = \frac{2}{\sqrt{3}} \tan mh\sqrt{3}, \tag{VI}$$

$$\sqrt{3} \sinh 3mh \cosh 2m\alpha = \sinh mh\sqrt{3}\left(\cosh 2mh + \frac{2\cosh mh}{\cos mh\sqrt{3}}\right),$$

$$\sqrt{3} \sinh 3mh \sinh 2m\alpha = \sinh mh\sqrt{3}\left(\sinh 2mh - \frac{2\sinh mh}{\cos mh\sqrt{3}}\right),$$

whence, eliminating α by squaring and subtracting, we obtain the period equation

$$\frac{3\sinh^2 3mh}{\sinh^2 mh\sqrt{3}} = \left(\cosh 2mh + \frac{2\cosh mh}{\cos mh\sqrt{3}}\right)^2 - \left(\sinh 2mh - \frac{2\sinh mh}{\cos mh\sqrt{3}}\right)^2$$
$$= 1 + \frac{4\cosh 3mh}{\cos mh\sqrt{3}} + \frac{4}{\cos^2 mh\sqrt{3}};$$

equivalent to

$$\cosh 3mh = -\cos mh\sqrt{3} + 2\sec mh\sqrt{3},$$

or

$$3\cosh 3mh = -\cos mh\sqrt{3} - 2\sec mh\sqrt{3},$$

the new period equations, which by inspection have an infinite number of real roots.

Transfer the axis of x to the edge of the water on one bank, and expand w in descending powers of e^{mh}, retaining only the leading terms, supposing h is made infinite; we shall thus obtain Kirchoff's expression for the motion of standing waves on a shore sloping at 30° to the horizon (Gesammelte Abhandlungen, II, p. 434).

To do this we must begin by writing $y - h\sqrt{3}$ for y, and $z + h$ for z; and then

$$\begin{aligned}\Phi &= \sinh m(z + h - a)\cos m(y - h\sqrt{3}) \\ &\quad - \sinh \tfrac{1}{2} m(y\sqrt{3} + z - 2h + 2a)\cos \tfrac{1}{2} m(y - z\sqrt{3} - 2h\sqrt{3}) \\ &\quad + \sinh \tfrac{1}{2} m(y\sqrt{3} - z - 4h - 2a)\cos \tfrac{1}{2} m(y + z\sqrt{3}).\end{aligned}$$

Now, when h is indefinitely great, the period equation gives $\sec mh\sqrt{3} = \infty$, $\cos mh\sqrt{3} = 0$; so that

$$\begin{aligned}\Phi &= \sinh m(z + h - a)\sin my \\ &\quad - \sinh \tfrac{1}{2} m(y\sqrt{3} + z - 2h + 2a)\sin \tfrac{1}{2} m(y - z\sqrt{3}) \\ &\quad + \sinh \tfrac{1}{2} m(y\sqrt{3} - z - 4h - 2a)\cos \tfrac{1}{2} m(y + z\sqrt{3}).\end{aligned}$$

Also, when $h = \infty$, $ml = 1$, and therefore $\tanh m(h - a) = 1$, so that

$$\cosh m(h - a) = \sinh m(h - a) = \tfrac{1}{2}\exp m(h - a) = C, \text{ suppose};$$

and, therefore, retaining the leading terms,

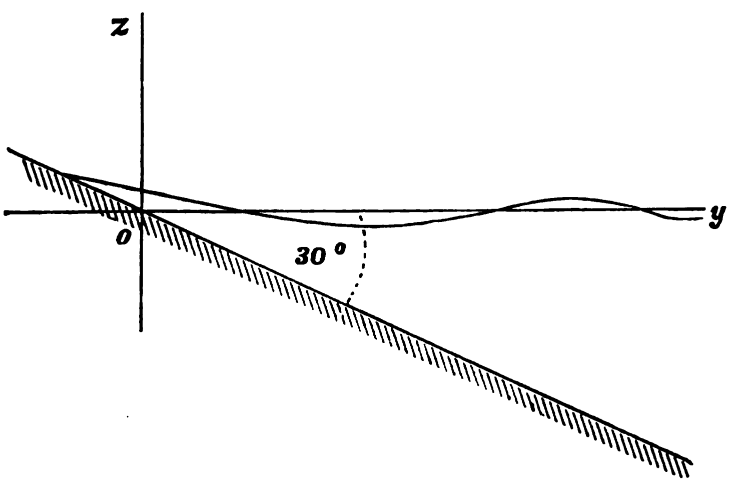
$$\begin{aligned}\Phi &= Be^{mz}\sin my \\ &\quad - Be^{-\frac{1}{2}m(y\sqrt{3}+z)}\sin \tfrac{1}{2} m(y - z\sqrt{3}) \\ &\quad + 2Be^{-m(h+2a)}e^{-\frac{1}{2}m(y\sqrt{3}-z)}\cos \tfrac{1}{2} m(y + z\sqrt{3}).\end{aligned}$$

But from equation (IV)

$$\sin mh\sqrt{3} = 1 = \sqrt{3}\sinh m(h + 2a) = \sqrt{3}\exp m(h + 2a),$$

so that we may replace

$$2e^{--m(h+2a)} \text{ by } \sqrt{3},$$



and then the value of Φ agrees with that given by Kirchoff, changing the sign of z, as our axis of z is drawn vertically upwards.

Then $$\Psi = -Be^{mz}\cos my + Be^{-\frac{1}{2}m(y\sqrt{3}+z)}\cos\frac{1}{2}m(y - z\sqrt{3})$$
$$+ B\sqrt{3}e^{-\frac{1}{2}m(y\sqrt{3}-z)}\sin\frac{1}{2}m(y + z\sqrt{3});$$
so that
$$\Phi + i\Psi = B\exp\left(mz - \frac{1}{2}i\pi\right) + B\sqrt{3}\exp(m\beta u - i\pi) + B\exp\left(m\beta^2 u - \frac{3}{2}i\pi\right).$$

To verify that in this case, where $z = 0$,
$$\frac{\partial\Phi}{\partial z} = m\Phi,$$
it is convenient to notice that $\frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial y}$; so that
$$\frac{\partial\Psi}{\partial y} = m\Phi,$$
where $z = 0$; and we may put $z = 0$ before differentiating with respect to y, which simplifies the work.

20. *Progressive Waves in a Channel of* 120°.

Just as from Kelland's solution for progressive waves in a channel of 90° we obtained the solution for standing waves across the channel by replacing the hyperbolic functions partially by circular functions, so, conversely, from the above solution for standing waves across a channel of 120°, we shall obtain the

solution for progressive waves by replacing the circular functions by the corresponding hyperbolic functions. Then, for progressive waves, we can put

$$\phi = A\Phi \cos\sqrt{2}\,(mx - nt),$$

where

$$\Phi = \sinh m(z-a)\cosh my - \sinh\tfrac{1}{2}m(y\sqrt{3}+z+2a)\cosh\tfrac{1}{2}m(y-z\sqrt{3})$$
$$+\sinh\tfrac{1}{2}m(y\sqrt{3}-z-2a)\cosh\tfrac{1}{2}m(y+z\sqrt{3}),$$

and then

$$\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=2\Phi;$$

so that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0;$$

so that the equation of continuity is satisfied.

We shall also find that when

(I) $$y = z\sqrt{3},\ \frac{\partial\Phi}{\partial y}=\sqrt{3}\,\frac{\partial\Phi}{\partial z},$$

when

(II) $$y = -z\sqrt{3},\ \frac{\partial\Phi}{\partial y}=-\sqrt{3}\,\frac{\partial\Phi}{\partial z};$$

so that the boundary conditions are satisfied.

To satisfy the free surface conditions, it is convenient to express Φ in the equivalent form

$$\Phi = \sinh m(z-a)\cosh my - \sinh\tfrac{1}{2}m\{(\sqrt{3}+1)y+2a\}\cosh\tfrac{1}{2}m(\sqrt{3}-1)y$$
$$+\sinh\tfrac{1}{2}m\{(\sqrt{3}-1)z-2a\}\cosh\tfrac{1}{2}m(\sqrt{3}+1)y;$$

and then the free surface conditions that

$$l\frac{\partial\Phi}{\partial z}=\Phi,$$

where $z=h$, for all values of y, is satisfied if

$$\begin{aligned} ml &= \tanh m(h-a)\\ &= (\sqrt{3}+1)\tanh\tfrac{1}{2}m\{(\sqrt{3}-1)h-2a\}\\ &= (\sqrt{3}-1)\tanh\tfrac{1}{2}m\{(\sqrt{3}+1)h+2a\};\end{aligned}$$

or, putting $m(h-a)=\gamma$,

$$\begin{aligned} ml = \tan h\gamma &= (\sqrt{3}+1)\tanh\left\{\gamma-\tfrac{1}{2}(3-\sqrt{3})mh\right\}\\ &= (\sqrt{3}-1)\tanh\left\{\tfrac{1}{2}(3+\sqrt{3})mh-\gamma\right\}.\end{aligned}$$

From these conditions we find that

$$2 \coth \gamma = \coth \tfrac{1}{2}(3 - \surd 3)\, mh + \coth \tfrac{1}{2}(3 + \surd 3)\, mh,$$

or $\tanh \gamma$ is an harmonic mean between

$$\tanh \tfrac{1}{2}(3 - \surd 3)\, mh \text{ and } \tanh \tfrac{1}{2}(3 + \surd 3)\, mh,$$

and by elimination of γ we obtain the period equation

$$\left\{\coth \tfrac{1}{2}(3-\surd 3)mh + \coth \tfrac{1}{2}(3+\surd 3)mh\right\}^2$$
$$+\surd 3\left\{\coth^2 \tfrac{1}{2}(3-\surd 3)\, mh - \coth^2 \tfrac{1}{2}(3+\surd 3)\, mh\right\} = 4,$$

equivalent to

$$(2-\surd 3)\cosh(3+\surd 3)\, mh + (2+\surd 3)\cosh(3-\surd 3)\, mh - \cosh(2mh\surd 3) - 3 = 0.$$

But on investigation it will be found that the only real root of this equation in mh is $mh = 0$, repeated four times, so that progressive waves of this type in a channel of 120° are unstable.

To represent such an unstable state of wave motion, let us change all the hyperbolic functions in Φ into the corresponding circular functions, and put

$$\phi = \Phi \cosh \surd 2\,(mx - nt),$$

in order that the equation of continuity may be satisfied.

Then we have

$$\Phi = \sin m(z - a) \cos my - \sin \tfrac{1}{2} m(y\surd 3 + z + 2a) \cos \tfrac{1}{2} m(y - z\surd 3)$$
$$+ \sin \tfrac{1}{2} m(y\surd 3 - z - 2a) \cos \tfrac{1}{2} m(y + z\surd 3);$$

so that

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = -2\Phi.$$

Then we shall find

(I) where $y = z\surd 3$, $\dfrac{\partial\Phi}{\partial y} = \surd 3\, \dfrac{\partial\Phi}{\partial z}$;

(II) where $y = -z\surd 3$, $\dfrac{\partial\Phi}{\partial y} = -\surd 3\, \dfrac{\partial\Phi}{\partial z}$;

so that the boundary conditions are satisfied.

Also, the conditions at the free surface $z = h$ are satisfied if

$$\begin{aligned} ml &= \tan m(h - a) \\ &= (\surd 3 + 1) \tan \tfrac{1}{2} m\{(\surd 3 - 1)h - 2a\} \\ &= (\surd 3 - 1) \tan \tfrac{1}{2} m\{(\surd 3 + 1)h + a\}; \end{aligned}$$

or,
$$ml = \tan\gamma$$
$$= (\sqrt{3}+1)\tan\left\{\gamma - \frac{1}{2}(3-\sqrt{3})\,mh\right\}$$
$$= (\sqrt{3}-1)\tan\left\{\frac{1}{2}(3+\sqrt{3})\,mh - \gamma\right\},$$
if we write Φ in the form
$$\Phi = \sin m(z-a)\cos my$$
$$+ \sin\frac{1}{2}m\{(\sqrt{3}-1)z - 2a\}\cos\frac{1}{2}m(\sqrt{3}+1)y$$
$$- \sin\frac{1}{2}m\{(\sqrt{3}+1)z + 2a\}\cos\frac{1}{2}m(\sqrt{3}-1)y.$$

As before, we shall find that $\tan\gamma$ is an harmonic mean between
$$\tan\frac{1}{2}(3-\sqrt{3})\,mh \text{ and } \tan\frac{1}{2}(3+\sqrt{3})\,mh,$$
and that the period equation is
$$\sqrt{3}\left\{\cot^2\frac{1}{2}(3-\sqrt{3})\,mh - \cot^2\frac{1}{2}(3+\sqrt{3})\,mh\right\}$$
$$-\left\{\cot\frac{1}{2}(3-\sqrt{3})\,mh + \cot\frac{1}{2}(3+\sqrt{3})\,mh\right\}^2 = 4,$$
or
$$(2-\sqrt{3})\cos(3+\sqrt{3})\,mh + (2+\sqrt{3})\cos(3-\sqrt{3})\,mh - \cos(2mh\sqrt{3}) - 3 = 0,$$
an equation with an infinite number of real roots.

This last value of Φ will be useful in attempting to investigate the *bore* or tidal *wave* in a river.

21. *General Wave Motion across a Channel with Plane Sides Sloping at Any Angle.*

Putting, as before,
$$u = z + iy = r(\cos\vartheta + i\sin\vartheta), \text{ and } w = \phi + i\psi,$$
and supposing the sides to slope equally at an angle $\alpha = \pi/2n$ to the horizon, let us attempt the general solution by putting
$$w = P_0\cos(u+\alpha_0) + P_1\cos(\beta_u + \alpha_1)$$
$$+ P_2\cos(\beta_u^2 + \alpha_2) + \ldots + P_{2n-1}\cos(\beta_u^{2n-1} + \alpha_{2n-1}),$$
where $\beta^{2n} = -1$, so that $\beta = e^{i\alpha}$, $\beta^n = i$.

Then we must determine the P's and α's, so as to satisfy the boundary conditions that $\psi = 0$ or w is real at the sides $\vartheta = \pm(n-1)\alpha$. The form will be different according as n is odd or even.

A. When n is odd, we must have the P's all real, and

$$P_0 = P_2 = P_4 \quad = \ldots. = A, \text{ suppose};$$
$$P_1 = P_3 = P_5 \quad = \ldots. = B, \text{ suppose};$$

also,

$$\alpha_0 = -\alpha_2 = \alpha_4 = -\alpha_6 = \ldots. = \gamma, \text{ a real quantity};$$
$$\alpha_1 = -\alpha_3 = \alpha_5 = -\alpha_7 = \ldots. = i\delta, \text{ an imaginary.}$$

Hence

$$w = A\{\cos(u+\gamma) + \cos(\beta^2 u - \gamma) + \cos(\beta^4 u + \gamma) + \ldots.\}$$
$$+ B\{\cos(\beta u + i\delta) + \cos(\beta^3 u - i\delta) + \cos(\beta^5 u + i\delta) + \ldots.\};$$

or, as it may be written,

$$w = A\{\cos(u+\gamma) + \cos(\beta^2 u - \gamma) + \cos(\beta^4 u + \gamma) + \ldots.\}$$
$$+ B\{\cosh(u+\delta) + \cosh(\beta^2 u - \delta) + \cosh(\beta^4 u + \delta) + \ldots.\},$$

involving, however, only three disposable constants, A/B, γ and δ.

When $n = 3$, this agrees with the expressions previously obtained, and the three equations of condition at the free surface gave the period equation for mh, the equation for l, and the equation for γ or δ, with A or B alternately zero.

But if we attempt to satisfy the conditions at the free surface with $n = 5, 7, \ldots.$, we shall have more equations to satisfy than the disposable constants mh, ml, A/B, γ and δ, so that the free surface conditions cannot be satisfied.

Separating w into its real and imaginary parts, we find

$$\phi = A\cos(z+\gamma)\cosh y + B\cosh(z+\delta)\cos y$$
$$+ A\cot(z\cos 2\alpha - y\sin 2\alpha - \gamma)\cosh(z\sin 2\alpha + y\cos 2\alpha)$$
$$+ B\cosh(z\cos 2\alpha - y\sin 2\alpha - \delta)\cos(z\sin 2\alpha + y\cos 2\alpha);$$

. . . .

$$\psi = -A\sin(z+\gamma)\sinh y + B\sinh(z+\delta)\sin y$$
$$- A\sin(z\cos 2\alpha - y\sin 2\alpha - \gamma)\sinh(z\sin 2\alpha + y\cos 2\alpha)$$
$$+ B\sinh(z\cos 2\alpha - y\sin 2\alpha - \delta)\sin(z\sin 2\alpha + y\cos 2\alpha).$$

B. When n is even, the boundary conditions lead to

$$P_0 = P_4 = P_8 = \ldots. = A + iB,$$
$$P_2 = P_6 = P_{10} = \ldots. = A - iB,$$
$$P_1 = P_5 = P_9 = \ldots. = C,$$
$$P_3 = P_7 = P_{11} = \ldots. = D,$$

and all the α's must vanish. Then

$$\begin{aligned} w = {} & (A + iB)(\cos u + \cos \beta^4 u + \cos \beta^8 u + \dots) \\ & + (A - iB)(\cos \beta^2 u + \cos \beta^6 u + \cos \beta^{10} u + \dots) \\ & + C(\cos \beta u + \cos \beta^5 u + \cos \beta^9 u + \dots) \\ & + D(\cos \beta^3 u + \cos \beta^7 u + \cos \beta^{11} u + \dots), \end{aligned}$$

involving three arbitrary constants, the ratios of A, B, C, D, so that we have not enough disposable constants to satisfy the free surface conditions for an even number greater than 2.

We may suppose the sides of the channel to slope to the horizon at any angles which are the same or different multiples of an n^{th} part of a right angle, and determine the P's and α's from the above general expression for w; thus, for a channel the sides of which slope at 60° to the horizon we shall find

$$\begin{aligned} w = {} & A\{\cos(u + i\gamma) + \cos(\beta^2 u - i\gamma) + \cos(\beta^4 u + i\gamma)\} \\ & + B\{\cosh(u + i\delta) + \cosh(\beta^2 u - i\delta) + \cosh(\beta^4 u + i\delta)\}, \end{aligned}$$

where $\beta^6 = -1$; but in this, as in the other cases, the free surface conditions cannot be satisfied.

Again, if one side slopes at 30° and the other at 60°, we shall find

$$\begin{aligned} w = {} & (A + iB)\cos u + E\cos \beta u + (A - iB)\cos \beta^2 u \\ & + (C + iD)\cos \beta^3 u + F\cos \beta^4 u + (C - iD)\cos \beta^5 u. \end{aligned}$$

For, if $\vartheta = 2\alpha$, where $\alpha = \frac{1}{6}\pi$, then $u = r\beta^2$, and

$$\begin{aligned} w = {} & (A + iB)\cos r\beta^2 + E\cos r\beta^3 + (A - iB)\cos r\beta^4 \\ & + (C + iD)\cos r\beta^5 + F\cos r\beta^6 + (C - iD)\cos r\beta^7, \end{aligned}$$

which is real, since β^2 and $-\beta^4$, β^5 and β^7 are conjugate imaginaries.

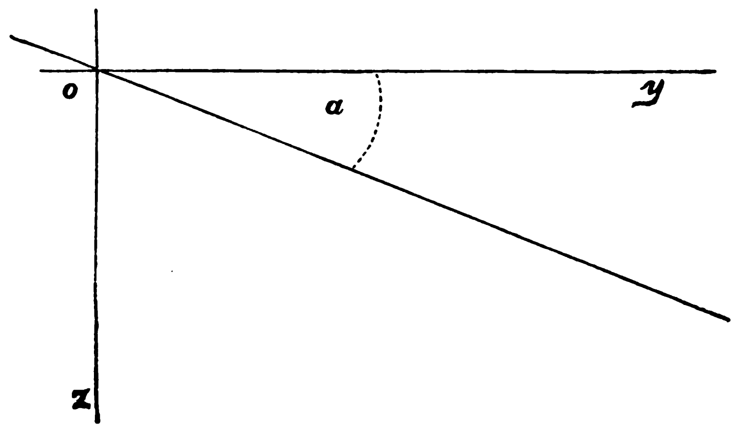
Again, if $\vartheta = -\alpha$, $u = r\beta^{-1}$, and

$$\begin{aligned} w = {} & (A + iB)\cos r\beta^{-1} + E\cos r + (A - iB)\cos r\beta \\ & + (C + iD)\cos r\beta^2 + E\cos r\beta^3 + (C - iD)\cos r\beta^4, \end{aligned}$$

which also is real, and therefore $\psi = 0$.

22. *Waves against a Uniformly Sloping Shore.*

Let us attempt in a similar manner to determine Kirchoff's general expressions for wave motion against a beach sloping uniformly at an angle α to the horizon (Gesammelte Abhandlungen, p. 431).



Suppose, now, that the axis of z is drawn vertically downwards, and let

$$u = y + iz = r(\cos\vartheta + i\sin\vartheta),$$

so that $\vartheta = a$ at the surface of the shore.

If we put

$$w = \phi + i\psi = \sum_{p=0}^{p=2n-1} P_p \exp i(\beta^p u + a_p),$$

where $a = \pi/2n$, then we must have $\psi = 0$, and therefore w real, when $\vartheta = a$, and consequently $u = r\beta$, where $\beta^{2n} = -1$, $\beta^n = i$. Then

$$\begin{aligned} w &= \Sigma P_p \exp i(r\beta^{p+1} + a_p) \\ &= \Sigma P_p \exp(r\beta^{n+p+1} + ia_p), \end{aligned}$$

and for this to be real, we must have $P_{2n-1} = 0$, and

P_0 and P_{2n-2} conjugate imaginaries, as also ia_0 and ia_{2n-2},
P_1 " P_{2n-3} " " " ia_1 " ia_{2n-3},
P_2 " P_{2n-4} " " " ia_2 " ia_{2n-4},
. . . .

P_{n-1} is real, and also ia_{n-1}.

We may then write $P_0 \exp ia_0$ and $P_{2n-2}\exp ia_{2n-2}$ in the form $A_0 \exp i\gamma_0$ and $A_0 \exp(-i\gamma_0)$ without loss of generality, and similar expressions for $P_1 \exp ia_1$, ... , and replace $P_{n-1}\exp ia_{n-1}$ by A_{n-1}, so that now

$$\begin{aligned} w = \sum_{p=0}^{p=n-2} A_p \,[&\exp\{-r\sin(\theta + pa) + ir\cos(\vartheta + pa) + i\gamma_p\} \\ &+ \exp\{-r\sin(\vartheta + 2n - p - 2\,.\,a) + ir\cos(\vartheta + 2n - p - 2\,.\,a) - i\gamma_p\}] \\ &+ A_{n-1}\exp\{-r\sin(\vartheta + n - 1\,.\,a) + ir\cos(\vartheta + n - 1\,.\,a)\}, \end{aligned}$$

giving ϕ and ψ, satisfying the equation of continuity, and satisfying the boundary condition that $\psi = 0$ and w therefore real when $\vartheta = \alpha = \pi/2n$.

At the free surface $\vartheta = 0$ we must have

$$l\frac{\partial\phi}{\partial z} = -\phi, \text{ or } l\frac{\partial\phi}{r\partial\vartheta} = -\phi,$$

for all values of r.

But
$$\frac{\partial\phi}{r\partial\vartheta} = -\frac{\partial\psi}{\partial z},$$

so that we must have
$$l\frac{\partial\psi}{\partial r} = \phi$$

for all values of r at the free surface $\vartheta = 0$; and we may put $\vartheta = 0$ in ϕ and ψ before differentiation, which simplifies the calculations considerably.

Now, putting $\vartheta = 0$,

$$\begin{aligned}
\phi = \sum_{p=0}^{p=n-2} A_p \{&\exp(-r\sin p\alpha)\cos(r\cos p\alpha + \gamma_p) \\
&+ \exp(-r\sin \overline{p+2}.\alpha)\cos(r\cos \overline{p+2}.\alpha + \gamma_p)\} \\
&+ A_{n-1}\exp(-r\sin \overline{n-1}.\alpha)\cos(r\cos \overline{n-1}.\alpha),
\end{aligned}$$

$$\begin{aligned}
\psi = \Sigma A_p \{&\exp(-r\sin p\alpha)\sin(r\cos p\alpha + \gamma_p) \\
&- \exp(-r\sin \overline{p+2}.\alpha)\sin(r\cos \overline{p+2}.\alpha + \gamma_p)\} \\
&+ A_{n-1}\exp(-r\sin \overline{n-1}.\alpha)\sin(r\cos \overline{n-1}.\alpha),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\psi}{\partial r} = \Sigma A_p \{&\exp(-r\sin p\alpha)\cos(r\cos p\alpha + p\alpha + \gamma_p) \\
&- \exp(-r\sin \overline{p+2}.\alpha)\cos(r\cos \overline{p+2}.\alpha + \overline{p+2}.\alpha + \gamma_p)\} \\
&+ A_{n-1}\exp(-r\sin \overline{n-1}.\alpha)\cos(r\cos \overline{n-1}.\alpha + \overline{n-1}.\alpha).
\end{aligned}$$

Equating coefficients in ϕ and $\frac{\partial\psi}{\partial r}$ of $\exp(-r\sin p\alpha)$, we have

$$\begin{aligned}
&A_0\cos(r + \gamma_0) \\
&= A_0\cos(r + \gamma_0), \\
&A_1\cos(r\cos\alpha + \gamma_1) \\
&= A_1\cos(r\cos\alpha + \alpha + \gamma_1), \\
&A_2\cos(r\cos 2\alpha + \gamma_2) + A_0\cos(r\cos 2\alpha + \gamma_0) \\
&= A_2\cos(r\cos 2\alpha + 2\alpha + \gamma_2) - A_0\cos(r\cos 2\alpha + 2\alpha + \gamma_0), \\
&A_3\cos(r\cos 3\alpha + \gamma_3) + A_1\cos(r\cos 3\alpha + \gamma_1) \\
&= A_3\cos(r\cos 3\alpha + 3\alpha + \gamma_3) - A_1\cos(r\cos 3\alpha + 3\alpha + \gamma_1), \\
&A_4\cos(r\cos 4\alpha + \gamma_4) + A_2\cos(r\cos 4\alpha + \gamma_2) \\
&= A_4\cos(r\cos 4\alpha + 4\alpha + \gamma_4) - A_2\cos(r\cos 4\alpha + 4\alpha + \gamma_2),
\end{aligned}$$

. . . .

$$A_{n-1}\cos(r\cos \overline{n-1}.a) + A_{n-3}\cos(r\cos \overline{n-1}.a + \gamma_{n-3})$$
$$= A_{n-1}\cos(r\cos \overline{n-1}.a + \overline{n-1}.a) - A_{n-3}\cos(r\cos \overline{n-1}.a + \overline{n-1}.a + \gamma_{n-3}),$$
$$A_{n-2}\cos\gamma_{n-2}$$
$$= -A_{n-2}\cos(na + \gamma_{n-2})$$
$$= \quad A_{n-2}\sin\gamma_{n-2}.$$

The first equation is satisfied identically, and the second is satisfied by $A_1 = 0$, which makes all the odd A's vanish, and then the even A's are determined by

$$A_2 \sin(r\cos 2a + a + \gamma_2)\sin a = A_0 \sin\left(r\cos 2a + a + \gamma_0 - \frac{1}{2}\pi\right)\cos a,$$

$$A_4 \sin(r\cos 4a + 2a + \gamma_4)\sin 2a = A_2 \sin\left(r\cos 4a + 2a + \gamma_2 - \frac{1}{2}\pi\right)\cos 2a,$$

. . . .

giving
$$A_2 \sin a = A_0 \cos a,$$
$$A_4 \sin 2a = A_2 \cos 2a,$$
. . . .

and
$$\gamma_2 = \gamma_0 - \frac{1}{2}\pi,$$
$$\gamma_4 = \gamma_2 - \frac{1}{2}\pi = \gamma_0 - \pi.$$
. . . .

The form of the result is different according as n is odd or even.

A. If n is odd $= 2m + 1$, then the final equations are

$$A_{2m}\cos(r\cos 2ma) + A_{2m-2}\cos(r\cos 2ma + \gamma_{2m-2})$$
$$= A_{2m}\cos(r\cos 2ma + 2ma) - A_{2m-2}\cos(r\cos 2ma + 2ma + \gamma_{2m-2});$$

or,

$$A_{2m}\sin(r\cos 2ma + ma)\sin ma = A_{2m-2}\sin\left(r\cos 2ma + ma + \gamma_{2m-2} - \frac{1}{2}\pi\right)\cos ma,$$

giving
$$A_{2m}\sin ma = A_{2m-2}\cos ma,$$

and
$$\gamma_{2m-2} = \frac{1}{2}\pi;$$

also,
$$A_{n-2} = A_{2m-1} = 0.$$

Then
$$\gamma_{2m-4} = \pi,$$
$$\gamma_{2m-6} = \frac{3}{2}\pi,$$
$$\gamma_{2m-8} = 2\pi,$$
. . . .

B. If n is even, $= 2m$, then A_{n-2} does not vanish; so that $\tan\gamma_{n-2} = 1$, $\gamma_{n-2} = \frac{1}{4}\pi$, whence the other values of γ are determined.

Suppose, for instance, $n=4$; then $\alpha=\frac{1}{8}\pi$, and

$$\gamma_2=\frac{1}{4}\pi,\ \gamma_0=\frac{3}{2}\pi;$$

also,
$$A_2=A_0\cot\alpha=A_0(\sqrt{2}+1),$$
agreeing with Kirchoff's results.

Again, suppose $n=5$, $\alpha=\frac{1}{10}\pi$;

then
$$\gamma_2=\frac{1}{2}\pi,\ \gamma_0=\pi;$$

also,
$$A_2=A_0\cot\alpha,$$
$$A_4=A_2\cot 2\alpha=A_0\cot\alpha\cot 2\alpha.$$

For
$$n=2,\ \gamma_0=\frac{1}{4}\pi,$$

and for
$$n=3,\ \gamma_0=\frac{1}{2}\pi,\ A_2=A_0\cot\alpha=A_0\sqrt{3},$$

agreeing with the results given by Kirchoff (Gesammelte Abhandlungen, p. 434).

23. *Algebraical Solution of Waves against a Sloping Beach.*

We can satisfy the equation of continuity and the boundary condition that $\psi=0$ when $\vartheta=\alpha=\pi/2n$ by putting

$$\begin{aligned}\psi+i\phi=&\,r^n(\cos n\vartheta+i\sin n\vartheta)\\ &+A_1ar^{n-1}\{\cos(n-1.\vartheta+\alpha)+i\sin(n-1.\vartheta+\alpha)\},\\ &+A_2a^2r^{n-2}\{\cos(n-2.\vartheta+2\alpha)+i\sin(n-2.\vartheta+2\alpha)\},\\ &+\ldots\ldots\ldots\ldots\ldots\\ &+A_na^n(\cos n\alpha+i\sin n\alpha),\end{aligned}$$

equivalent to

$$\psi+i\phi=u^n+A_1a\beta u^{n-1}+A_2a^2\beta^2u^{n-2}+\ldots.,$$

an algebraical function of

$$u=y+iz=r(\cos\vartheta+i\sin\vartheta);$$

and then $\psi=0$ when $\vartheta=\alpha$, since $n\alpha=\frac{1}{2}\pi$.

Put
$$\begin{aligned}\psi=&\,r^n\cos n\vartheta-A_1ar^{n-1}\cos(n-1.\vartheta+\alpha)\\ &+A_2a^2r^{n-2}\cos(n-2.\vartheta+2\alpha)-A_3a^3r^{n-3}\cos(n-3.\vartheta+3\alpha)\\ &\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

and then at the free surface $\vartheta=0$ we shall have

$$a\frac{\partial\psi}{\partial r}=\phi$$

for all values of r, if

$$\begin{aligned} A_1 \sin \alpha &= n, \\ A_2 \sin 2\alpha &= (n-1) A_1 \cos \alpha, \\ A_3 \sin 3\alpha &= (n-2) A_2 \cos 2\alpha, \\ &\cdots\cdots\cdots \\ A_n \sin n\alpha &= A_{n-1} \cos (n-1)\alpha; \end{aligned}$$

n equations for determining $A_1, A_2, A_3, \ldots. A_n$.

Here the motion increases indefinitely with y and z, so we must seek to determine possible boundaries to limit the motion, and to contain the liquid in a finite cylinder.

Suppose, for instance, $n = 3$; then for $\alpha = \frac{1}{6}\pi$,

$$A_1 = 6, \quad A_2 = 12, \quad A_3 = 6;$$

so that

$$\psi + i\phi = \left(r^{i\vartheta} - \frac{ae^{i\alpha}}{\sin \alpha}\right)^3 + \text{const.},$$

and with a new origin

$$\psi + i\phi = u^3 + \text{const.},$$

the algebraical motion previously investigated in a channel of 120°.

Again, suppose $n = 4$, $\alpha = \frac{1}{8}\pi$; then

$$A_1 \sin \alpha = 4, \quad A_2 \sin 2\alpha = 3A_1 \cos \alpha, \quad A_3 \sin 3\alpha = 2A_2 \cos 2\alpha, \quad A_4 = A_3 \cos 3\alpha,$$

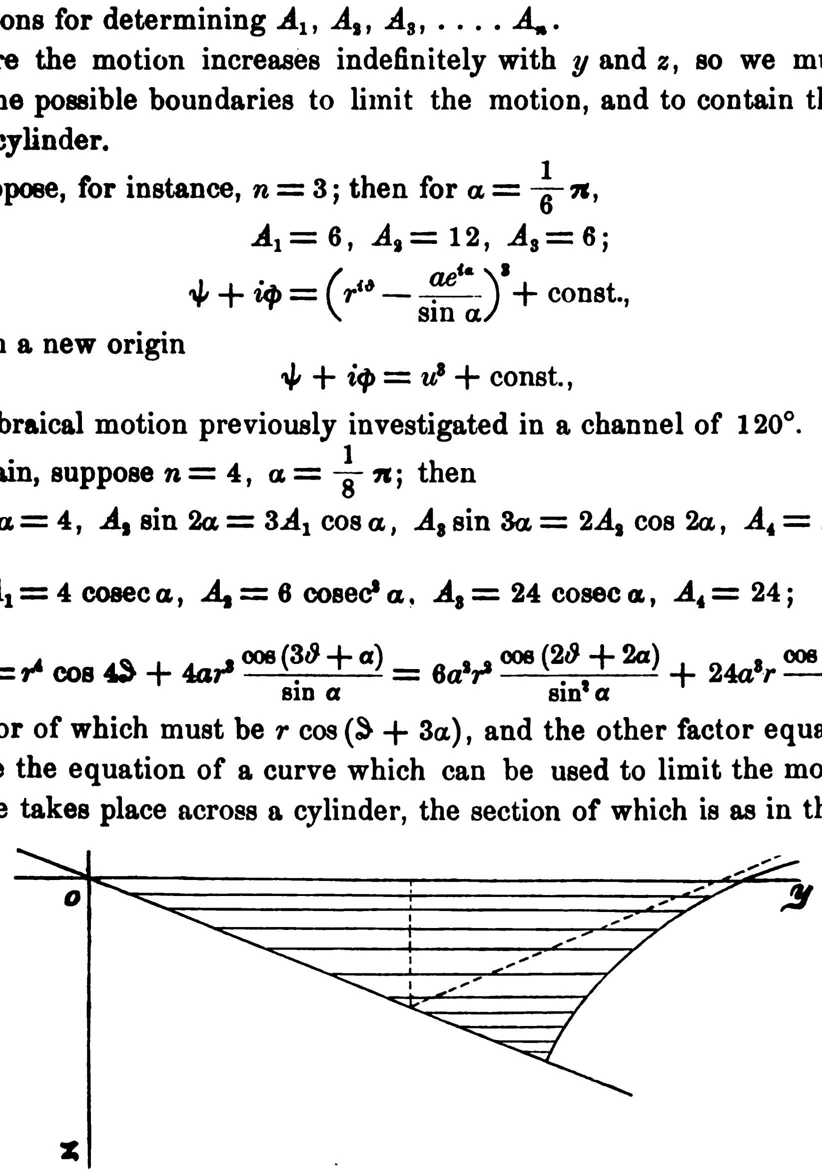
giving

$$A_1 = 4 \operatorname{cosec} \alpha, \quad A_2 = 6 \operatorname{cosec}^2 \alpha, \quad A_3 = 24 \operatorname{cosec} \alpha, \quad A_4 = 24;$$

so that

$$\psi = r^4 \cos 4\vartheta + 4ar^3 \frac{\cos(3\vartheta + \alpha)}{\sin \alpha} = 6a^2r^2 \frac{\cos(2\vartheta + 2\alpha)}{\sin^2 \alpha} + 24a^3 r \frac{\cos(\vartheta + 3\alpha)}{\sin \alpha},$$

one factor of which must be $r \cos(\vartheta + 3\alpha)$, and the other factor equated to zero will give the equation of a curve which can be used to limit the motion, which therefore takes place across a cylinder, the section of which is as in the figure.



24. *Wave Motion in a Cone.*

If we put $\phi = zr^n \sin n\vartheta$,

employing the cylindrical co-ordinates r, ϑ, z, then the equation of continuity

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{r^2\partial\vartheta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

is satisfied; also, at the free surface $z = h$,

$$h\frac{\partial\phi}{\partial z} = \phi;$$

so that the length of the equivalent simple pendulum of the wave motion is h.

We must now seek to determine the shape of a vessel which will contain liquid having the above given motion.

Supposing the vessel is of revolution, then along a meridian section, the axis being vertical,

$$\frac{dz}{\frac{\partial\phi}{\partial z}} = \frac{dr}{\frac{\partial\phi}{\partial r}}$$

or,
$$\frac{dz}{r^n \sin n\vartheta} = \frac{dr}{nr^{n-1} \sin n\vartheta};$$

or,
$$nzdz = rdr;$$

so that
$$nz^2 = r^2 + \text{const.}$$

is the equation of a meridian section, which is therefore in general a hyperbola, except when the constant is zero, when it degenerates into two straight lines.

The vessel is therefore a hyperboloid of revolution, including a cone of vertical angle $2\tan^{-1}\sqrt{n}$ as a particular case.

If $n = 1$, $\phi = yz$,

the same as for the algebraical motion across a channel of 90°.

The stream lines are rectangular hyperbolas, so that the boundary may be supposed a horizontal cylinder on a vertical cone of which the vertical sections are rectangular hyperbolas.

If $n = 2$, $\phi = xyz$,

and the vertical angle of the cone is $2\tan^{-1}\sqrt{z}$.

The differential equations of the stream lines are now

$$xdx = ydy = zdz,$$

so that we may suppose the containing vessel, in its most general form, a hyperboloid whose equation is

$$ax^2 + by^2 + cz^2 = \text{constant},$$

subject to the condition $a + b + c = 0$.

Generally, for any value of n, we may suppose vertical meridian plane diaphragms, given in position by

$$\cos n\vartheta = 0,$$

to be introduced without disturbing the motion.

Also, since $\frac{\partial^2\phi}{\partial z^2} = 0$, this kind of wave motion will be unaffected by any capillarity of the free surface.

25. *Wave Motion in a Cylinder.*

In seeking the particular solutions of the above equation of continuity in cylindrical co-ordinates, if we assume that ϕ has a factor $\sin n\vartheta$, so that the wave motion may be limited by vertical meridian diaphragms, given by $\cos n\vartheta = 0$, then

$$r^2\frac{\partial^2\phi}{\partial r^2} + r\frac{\partial\phi}{\partial r} + r^2\frac{\partial^2\phi}{\partial z^2} - n^2\phi = 0;$$

and if we further assume that ϕ has a factor $\cosh(kz+\beta)$, the other factor being a function of r only, then

$$r^2\frac{\partial^2\phi}{\partial r^2} + r\frac{\partial\phi}{\partial r} + (k^2r^2 - n^2)\phi = 0,$$

Bessel's differential equation; so that we may put

$$\phi = \cosh(kz+\beta)\,J_n(kr)\sin n\vartheta,$$

as a type of the particular solutions of the equation of continuity.

A single term of this nature is appropriate for determining wave motion inside a vertical circular cylinder.

Since $\frac{\partial\phi}{\partial z} = 0$ at the base, $z=0$, therefore $\beta = 0$; also, at the free surface, $z=h$,

$$l\frac{\partial\phi}{\partial z} = \phi,$$

or

$$kl = \coth kh;$$

and k is determined from the condition that at the cylindrical boundary $r=a$, $\frac{\partial\phi}{\partial r} = 0$, or

$$J_n'(ka) = 0.$$

This kind of wave motion in a cylinder has been completely investigated by Lord Rayleigh.

When $n = i + \frac{1}{2}$, where i denotes an integer, then the corresponding Bessel's function is an algebraical and trigonometrical function of r, so that we can obtain a corresponding solution for wave motion in the space bounded by the cylinder and two diametral planes inclined at an angle $2\pi/(2i+1)$; *e. g.* $\frac{2}{3}\pi, \frac{2}{5}\pi, \ldots$.

INDEX.

Lectures on the Theory of Reciprocants.

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[Reported by JAMES HAMMOND, M. A.]

LECTURE XVII.

The fundamental reciprocants for extent 3, given in the last lecture, agree with the irreducible invariants of a binary cubic both in number and type, with the single exception that the degree of the cubic discriminant is lower by unity than that of the reciprocant corresponding to it. When the extent is raised to 4, both the discriminant and its analogue cease to rank among the irreducible forms, the former being expressible as a rational integral function of invariants of lower degree, and the latter as a similar function of reciprocants. But the increase of extent introduces three additional reciprocants whose leading terms are a^2e, a^2ce and a^3a^2, whereas the additional invariants are only two in number and begin with ae and ace respectively.

The irreducible reciprocants of extent 4 are as follows:

deg. wt.

1.0 a,

2.2 $4M = 4ac - 5b^2$,

3.3 $A = a^2d - 3abc + 2b^3$,

3.4 $P_4 = 50a^2e - 175abd + 28ac^2 + 105b^2c$,*

4.6 $(a^2ce) = 800a^2ce - 1000ab^2e - 875a^2d^2 + 2450abcd - 1344ac^3 - 35b^2c^2$,

5.8 $(a^3e^2) = 625a^3e^2 - 4375a^2bde - 49700a^2c^2e + 128625ab^2ce - 78750b^4e$
$+ 55125a^2cd^2 - 61250ab^2d^2 - 156800abc^2d + 183750b^3cd$
$+ 84868ac^4 - 102165b^2c^3$.

---

* P_4 is the protomorph of minimum degree; the other protomorph, B, which will be used when we treat of Principiants, is, when expressed in terms of the irreducible forms,

$$B = \frac{1}{50}(aP_4 - 128M^2).$$

The similar list of invariants for the quartic is

deg. wt.	
1.0	a,
2.2	$ac - b^2$,
3.3	$a^2d - 3abc + 2b^3$,
2.4	$ae - 4bd + 3c^2$,
3.6	$ace - b^2e - ad^2 + 2bcd - c^3$.

To obtain the fundamental forms of extent 4 we have to combine M, A and the Quasi-Discriminant

$$(a^3d^2) = 125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c$$

with the additional Protomorph

$$P_4 = 50a^2e - 175abd + 28ac^2 + 105b^2c$$

in such a manner that the combination contains a factor a. The removal of this factor gives rise to a form of lower degree, and the process is repeated as often as possible.

Calling that portion of any form which does not contain a its residue, the residue of $4M$ is $-5b^2$, that of (a^3d^2) being $-300b^4c$, and that of P_4 being $105b^2c$. Thus

$$16MP_4 - 7(a^3d^2)$$

contains the factor a, and leads to (a^2ce) of the type 6; 4, 4, which is the analogue to the Catalecticant

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

The form (a^3d^2) now ceases to be a groundform (= irreducible form) and is replaced by the Quasi-Catalecticant (a^2ce), for

$$(a^3d^2) = \frac{16}{7}MP_4 - \frac{1}{7}a(a^2ce).$$

Similarly, the Cubic Discriminant, a groundform *quâ* the letters a, b, c, d, becomes reducible when a new letter, e, is introduced, and is then replaced by the Catalecticant.

We now come to an extra form which has no analogue in invariants. The residue of the Quasi-Catalecticant (a^2ce) is $-35b^2c^2$, and consequently

$$P_4^2 - 252M(a^2ce)$$

divides by a numerical multiple of a (as it happens by $4a$) and yields the form (a^3e^2), whose type is 8; 5, 4.

Here the deduction of new fundamental forms comes to an end on account of the appearance of e in the residue of (a^3e^2). It would have ended sooner but for the apparently accidental non-appearance of the term b^3d (of the same type 6; 4, 4 as b^2c^2) in the residue of (a^2ce). Had this term appeared, no combination could have been made leading to a new groundform after (a^2ce). We are able to show from *à priori* considerations that it cannot exist.

For the arguments in the annihilator V, up to ∂_e inclusive, are

$$a^2\partial_b,\ ab\partial_c,\ ac\partial_d,\ b^2\partial_d,\ ad\partial_e,\ \text{and}\ bc\partial_e.$$

If, now, the term μb^3d were to form part of a Pure Reciprocant, $b^2\partial_d$ operating upon it would give μb^5; but every other portion of the operator would necessarily give terms containing one or other of the letters a, c. Since such terms cannot destroy μb^5, we must have $\mu b^5 = 0$. Hence the term in question is necessarily non-existent.

The method of combining the protomorphs which we have followed shows that the fundamental reciprocants of extent 4 are connected *inter se* by the two relations or syzygies

$$7(256M^3 + 125A^2) - 16aMP_4 + a^2(a^2ce) = 0,$$
$$P_4^2 - 252M(a^2ce) - 4a(a^3e^2) = 0.$$

The invariants of the binary quartic are connected by only one syzygy, similar to the first of these; the second has no analogue in the theory of Invariants. It has been shown that the irreducible reciprocants of extent 3 are connected by the syzygy

$$256M^3 + 125A^2 - a(a^3d^2) = 0.$$

Substituting in this for the Quasi-Discriminant (a^3d^2) its value expressed in terms of the fundamental forms of extent 4, by means of the equation

$$16MP_4 - 7(a^3d^2) = a(a^2ce),$$

we obtain the first of the above syzygies. By a precisely similar substitution, the syzygy connecting the invariants of the quartic is derived from the one which connects the invariants of the cubic.

Every reciprocant of extent 4 is a rational integral function of the six fundamental forms given in the table; and, by means of the syzygies, powers, but not products, of A and P_4 can be removed from this function. For the first syzygy gives A^2 and the second gives P_4^2 as a rational integral function of the

four remaining forms a, M, (a^2ce), and (a^3e^2). Hence every reciprocant of extent 4 is of one or other of the forms

$$\Phi,\ A\Phi,\ P_4\Phi,\ AP_4\Phi,$$

where Φ does not contain either A or P_4, but is a rational integral function of the other four fundamental forms.

Let the four forms which appear in Φ occur raised to the powers κ, λ, μ, ν, respectively, in one of its terms. Since the degree-weights of these four forms are

$$1.0,\quad 2.2,\quad 4.6 \text{ and } 5.8,$$

any such term may be represented by

$$a^\kappa (a^2x^2)^\lambda (a^4x^6)^\mu (a^5x^8)^\nu.$$

Thus the totality of the terms in Φ will be represented by

$$\Sigma a^\kappa (a^2x^2)^\lambda (a^4x^6)^\mu (a^5x^8)^\nu = \frac{1}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}.$$

Now, A, P_4 and AP_4 have the degree-weights

$$3.3,\quad 3.4 \text{ and } 6.7,$$

and consequently the totality of terms in

$$\Phi,\ A\Phi,\ P_4\Phi \text{ and } AP_4\Phi$$

(*i. e.* the totality of the pure reciprocants of extent 4) will be represented by

$$(1 + a^3x^3 + a^3x^4 + a^6x^7)\, \Sigma a^\kappa (a^2x^2)^\lambda (a^4x^6)^\mu (a^5x^8)^\nu$$
$$= \frac{1 + a^3x^3 + a^3x^4 + a^6x^7}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}.$$

Hence the number of Pure Reciprocants of the type $w;\ i,\ 4$ is the coefficient of a^ix^w in the expansion of a fraction whose numerator is

$$1 + a^3x^3 + a^3x^4 + a^6x^7,$$

with the denominator

$$(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8).$$

This fraction is called the Representative Form of the Generating Function, in contradistinction to the Crude Form, which is a fraction with the numerator

$$1 - a^{-1}x,$$

having for its denominator

$$(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4).$$

The crude form expresses the fact that the number of pure reciprocants of the

type $$w;\ i,\ j$$
is $$(w;\ i,\ j) - (w-1;\ i+1,\ j).$$
Its numerator is $1 - a^{-1}x$ for all extents; for the general case in which the extent is j, its denominator consists of the $j+1$ factors
$$(1-a)(1-ax)(1-ax^2)\ldots.(1-ax^j).$$

The removal of the negative terms [corresponding to cases in which $(w; i, j) < (w-1;\ i+1, j)$] from the crude form would give either the representative form or one equivalent to it, according as the representative form is or is not in its lowest terms. In the parallel theory of Invariants the terms to be rejected are those for which $ij - 2w < 0$; but we do not at present know of any similar criterion for reciprocants, and are thus unable to pass directly from the crude to the representative form of their generating function.

Knowing both the crude and the representative form for reciprocants of extent 4, we may verify that the difference between these two forms of the generating function is omninegative. It will be found that
$$\frac{1-a^{-1}x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)}$$
$$= \frac{1+a^3x^3+a^3x^4+a^6x^7}{(1-a)(1-a^2x^2)(1-a^4x^6)(1-a^5x^8)}$$
$$- \frac{1}{(1-ax^2)(1-ax^3)(1-ax^4)}\left(\frac{a^{-1}x+a^2x^5}{1-a^4x^6}+\frac{x^2+a^2x^6}{1-a^5x^8}\right)$$
$$- \frac{1}{(1-ax^4)(1-a^4x^6)(1-a^5x^8)}\left(\frac{x+a^5x^{10}}{1-ax^2}+\frac{a^3x^5+a^2x^7}{1-ax^3}\right).$$
Thus the crude form is seen to consist of an omnipositive part, equal to the representative form, and an omninegative part.

There is no difficulty in obtaining the representative form of the generating function for pure reciprocants of extents 2 and 3. In the one case every reciprocant is a rational integral function of two forms of degree-weight, 1.0 and 2.2 respectively. The generating function is therefore
$$\frac{1}{(1-a)(1-a^2x^2)}.$$
In the other case (*i. e.* for extent 3) every pure reciprocant can be expressed as a rational integral function of four forms, of which the degree-weights are 1.0, 2.2, 3.3 and 5.6, no higher power than the first of the form 3.3 occurring in the function. Thus the representative form is
$$\frac{1+a^3x^3}{(1-a)(1-a^2x^2)(1-a^5x^6)}.$$

LECTURE XVIII.

The number of Pure Reciprocants of a given degree is finite; the number of Invariants of the same degree is infinite. Thus, for example, we have the well-known series of invariants

$$ac - b^2, \ ae - 4bd + 3c^2, \ \ldots\ldots,$$

all of degree 2, but of weights and extents proceeding to infinity. This may be proved from the theory of partitions (see *American Journal of Mathematics*, Vol. V, No. 1, On Subinvariants, Excursus on Rational Fractions and Partitions). It will be seen in that article that if $N(w:i)$ is the number of ways in which w can be divided into i parts, and if P is the least common multiple of $2, 3, 4, \ldots\ldots i$, then $N(w:i)$ can be expressed under the form

$$F(w, i) + F'(w, i, p),$$

where p is the residue of w in respect of P.

Writing $$w + \frac{i(i+1)}{4} = \nu,$$

$F(w, i)$ is of the form $$\frac{\nu^{i-1}}{2^2.3^2\ldots\ldots(i-1)^2.i} + \ldots\ldots,$$

all the succeeding indices of the powers of ν in $F(w, i)$ decreasing by 2, and their coefficients being transcendental functions of i which involve Bernoulli's Numbers.

In $F'(w, i, p)$ the highest index of ν is one unit less than the number of times that i is divisible by 2, *i. e.* is $\frac{i-2}{2}$ or $\frac{i-3}{2}$, according as i is even or odd.

Thus, for the partitions of w into 3 parts, we have the formula

$$N(w:3) = \left\{\frac{\nu^2}{12} - \frac{7}{72}\right\} + \left\{\frac{1}{8}(-1)^{\nu+1} + \frac{1}{9}(\rho_1^\nu + \rho_2^\nu)\right\},$$

where $\nu = w + \frac{1+2+3}{2} = w + 3$.

And, for the partitions of w into 4 parts,

$$N(w:4) = \left\{\frac{\nu^3}{144} - \frac{5\nu}{96}\right\} + \left\{\frac{1}{32}(-)^{\nu+1}\nu + \frac{1}{27}(\rho_1^{\nu+1} + \rho_2^{\nu+1} - \rho_1^{\nu-1} - \rho_2^{\nu-1}) \right.$$
$$\left. - \frac{1}{32}(i_1^{\nu+1} + i_2^{\nu+1} - i_1^{\nu-1} - i_2^{\nu-1})\right\},$$

where $\nu = w + \frac{1+2+3+4}{2} = w + 5$,

and ρ_1, ρ_2 are the roots of $\rho^2 + \rho + 1 = 0$,
i_1, i_2 " " " " $i^2 + 1 = 0$;

in other words, ρ_1 and ρ_2 are primitive cube roots, and i_1, i_2 primitive fourth roots of unity.

The principal term of $N(w:3)$, regarded as a function of w, is

$$\frac{w^2}{12} = \frac{w^2}{2^2.3}, \text{ that of } N(w:4) \text{ being } \frac{w^3}{144} = \frac{w^3}{2^2.3^2.4}.$$

And in general the principal term of $N(w:i)$ is

$$\frac{w^{i-1}}{2^2.3^2.4^2.\ldots.(i-1)^2.i}.$$

Hence it follows, from a general algebraical principle, that for all values of w above a certain limit, which depends on the value of i and may be determined by the aid of partition tables, $(w; i, \infty) - (w-1; i+1, \infty)$ must become negative.

Ultimately, $\dfrac{(w-1; i+1, \infty)}{(w; i, \infty)} = \dfrac{w}{i(i+1)}$, which must eventually be greater than unity. This shows that beyond a certain value of w there can be no pure reciprocant, and consequently that the number of pure reciprocants of a given degree i is finite.

Mr. Hammond remarks that the formulae for $N(w:3)$ and $N(w:4)$ may, by the substitution of trigonometrical expressions for the roots of unity, accompanied by some easy reductions, be transformed into

$$N(w:3) = \frac{\nu^2}{12} + \frac{1}{4}\sin^2\frac{\nu\pi}{2} - \frac{4}{9}\sin^2\frac{\nu\pi}{3}$$

and $$N(w:4) = \frac{\nu^3}{144} - \frac{\nu}{12} + \frac{\nu}{16}\sin^2\frac{\nu\pi}{2} + \frac{1}{8}\sin\frac{\nu\pi}{2} - \frac{2}{9\sqrt{3}}\sin\frac{\nu\pi}{3},$$

where, in the first formula, $\nu = w + 3$, and in the second $\nu = w + 5$. He also obtains the principal term of $N(w:i)$ from first principles as follows:

The partitions of w into i parts may be separated into two sets, the first containing at least one zero part in each of its partitions, the second consisting of partitions in which no zero part occurs.

Suppressing one zero part in each partition of the first set, we see that the number of partitions in which 0 occurs is $N(w:i-1)$. Diminishing each part by unity in those partitions which contain no zeros, their number is seen to be $N(w-i:i)$. The sum of these two numbers is $N(w:i)$, which is the total

number of partitions, and consequently $N(w:i) = N(w:i-1) + N(w-i:i)$. Let the principal term of $N(w:i-1)$ be aw^{i-2}, where a is independent of w, and write $w = ix,\ N(w:i) = u_x,\ N(w-i:i) = u_{x-1}$.

Then $u_x - u_{x-1} = aw^{i-2} + \ldots\ldots = ai^{i-2}x^{i-2} + \ldots\ldots$

Hence, by a simple summation, we find

$$u_x = ai^{i-2}\{x^{i-2} + (x-1)^{i-2} + (x-2)^{i-2} + \ldots\ldots\} + \ldots\ldots$$

But, since only the principal term of u_x is required, this summation may be replaced by an integration. Thus the principal term of u_x is

$$ai^{i-2}\int x^{i-2}dx = \frac{ai^{i-2}x^{i-1}}{i-1}.$$

Restoring $w = ix$ and $N(w:i) = u_x$,

we see that the principal term of $N(w:i)$ is $\dfrac{aw^{i-1}}{(i-1)i}$. Thus the principal term of $N(w:i)$ is found from that of $N(w:i-1)$ by multiplying it by $\dfrac{w}{(i-1)i}$.

When $i = 3$, the principal term is $\dfrac{w^2}{2^2.3}$; it is therefore $\dfrac{w^3}{2^2.3^2.4}$ when $i = 4$; and for the general case it is $\dfrac{w^{i-1}}{2^2.3^2.4^2.\ldots.(i-1)^2.i}$.

The value of $N(w:i)$ is given in line i and column w of the following table:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	2	2	3	3	4	4	5	5	6	6	7	7	8
3	1	2	3	4	5	7	8	10	12	14	16	19	21	24
4	1	2	3	5	6	9	11	15	18	23	27	34	39	47
5	1	2	3	5	7	10	13	18	23	30	37	47	57	70
6	1	2	3	5	7	11	14	20	26	35	44	58	71	90

From an inspection of the tabulated values of $N(w:i)$ we see that

$N(w:2) - N(w-1:3)$ is negative or zero when $w > 2$,
$N(w:3) - N(w-1:4)$ " " " " " $w > 6$,
$N(w:4) - N(w-1:5)$ " " " " " $w > 8$,
$N(w:5) - N(w-1:6)$ " " " " " $w > 12$.

Hence for pure reciprocants of indefinite extent, whose degrees are

2, 3, 4, 5,

the highest possible weights are 2, 6, 8 and 12, respectively.

In like manner, from Euler's table, in his memoir De Partitione Numerorum (published in 1750), it will be found that

for degrees	2	3	4	5	6	7	8	9	10	11	12	13
the highest weights are	2	6	8	12	16	21	26	30	36	42	49	55

Further than this the table, which goes up to $w = 59$, will not enable us to proceed.

The actual number of pure reciprocants of degree i, weight w, and of indefinite extent, is seen in the following table, which gives the value of $N(w:i) - N(w-1:i+1)$ when positive, blank spaces being left in the table when this difference is zero or negative.

Degree $i =$ \ Weight $w =$	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1												
3	1	1	1		1								
4	1	1	2	1	2	1	2						
5	1	1	2	2	3	2	4	3	4	2	3		

Thus, for degree 2, there is only one pure reciprocant, viz.

$$(ac) = 4ac - 5b^2.$$

For degree 3 the table shows that, in addition to the compound form

$$a(ac) = a(4ac - 5b^2),$$

there are three others whose weights are 3, 4 and 6 respectively.

These are the three protomorphs,

$$(a^2d) = a^2d - 3abc + 2b^3,$$
$$(a^2e) = 50a^2e - 175abd + 28ac^2 + 105b^2c,$$
$$(a^2g) = 14a^2g - 63abf - 1350ace + 1782b^2e + 1470ad^2 - 4158bcd + 2310c^3.$$

With the above forms and a we are able to form the following compounds

of degree 4:	$a^2(ac)$,	$a(a^2d)$,	$(ac)^2$,	$a(a^2e)$,	$a(a^2g)$,
whose weights are	2,	3,	4,	4,	6.

The forms of degree 4 and weights 5, 7, 8, and one of the forms of weight 6, cannot be similarly made up of forms of inferior degree, and are therefore groundforms. Three of them are the protomorphs (a^3f), (a^3h) and (a^3i) of weights 5, 7 and 8, whose values were given in Lecture XVI. The groundform of weight 6 is the Quasi-Catalecticant given in the last lecture. All the

forms of degree 4 have thus been accounted for except one of the two forms of weight 8, which will be seen to be of extent 6, and to have a^2cg for its leading term.

We know from Euler's table that $N(8:4) - N(7:5) = 2$; *i. e.*

$$(8;4,8) - (7;5,8) = 2.$$

Now,

$$(8;4,7) = N(8:4) - 1, \text{ the omitted partition being } 8.0.0.0,$$

$$(8;4,6) = N(8:4) - 2, \text{ the partition } 7.1.0.0 \text{ being also left out,}$$

$$(8;4,5) = N(8:4) - 4, \begin{cases} \text{for } 6.2.0.0 \text{ and } 6.1.1.0 \text{ are excluded from} \\ (8;4,5), \text{ but make their appearance in } (8;4,6). \end{cases}$$

Similarly,

$$(7;5,7) = N(7:5),$$
$$(7;5,6) = N(7:5) - 1,$$
$$(7;5,5) = N(7:5) - 2.$$

We have, therefore,

$$(8;4,8) - (7;5,8) = 2,$$
$$(8;4,7) - (7;5,7) = 1,$$
$$(8;4,6) - (7;5,6) = 1,$$
$$(8;4,5) - (7;5,5) = 0.$$

Hence we may draw the following inferences:

(1). No pure reciprocant exists whose type is 8; 4, 5.

(2). The one whose type is 8; 4, 6 must contain the letter g.

(3). No fresh form is found by making the extent 7 instead of 6, so that there is no pure reciprocant of weight 8 and degree 4 whose *actual extent* is 7.

(4). There is a pure reciprocant (the Protomorph whose leading term is a^3i) whose actual extent is 8.

(5). This, with the one whose actual extent is 6, makes up the two given by $(8;4,8) - (7;5,8) = 2$.

LECTURE XIX.

The following is a complete list of the irreducible reciprocants of indefinite extent for the degrees 2, 3 and 4:

Deg. wt.		
2.2	(ac),	
3.3	(a^2d),	
3.4	(a^2e),	
3.6	(a^2g),	
4.5	(a^3f),	
4.6	(a^2ce),	
4.7	(a^3h),	
4.8	(a^3i),	(a^2cg).

The values of all of them except (a^2cg) have been given in previous lectures, and the method of obtaining them sufficiently indicated. Thus (ac), (a^2d), (a^2e), (a^3f), (a^3g), (a^3h) and (a^3i) are the Protomorphs of minimum degree P_2, P_3, P_4, P_5, P_6, P_7 and P_8, respectively; and (a^2ce) is the Quasi-Catalecticant whose value has been set forth in the table of irreducible forms of extent 4. It will be remembered that (a^2ce) was found by combining the Quasi-Discriminant (a^3d^2) with P_2P_4 linearly in such a manner that the combination, which is of the 5th degree, divides by a and gives (a^2ce) of the 4th degree. If we try to find (a^2cg) by a similar process, it will be necessary to rise as high as the 7th degree, and then to drop down by successive divisions by a to the fourth.

In fact, since to a numerical factor près the residues of

	$P_2,\ P_3,\ P_4,\ P_5$
are	$b^2,\ b^3,\ b^2c,\ b^2c,$
that of	P_3P_5 will be b^6c,
and that of	$P_2^2P_4$ will be b^6c.

Thus a linear combination of P_3P_5 and $P_2^2P_4$ will be divisible by a, and, taking account of the numerical coefficients, we shall find

$$26P_2^2P_4 + 875P_3P_5 \equiv 0 \text{ (mod. } a).$$

As a result of calculation, it will be seen that the above combination of the protomorphs divided by a,

$$\frac{1}{a}(26P_2^2P_4 + 875P_3P_5),$$

has (to a numerical factor près) the same residue as P_4^2.

Making a second combination and division by a, we find

$$7\left(\frac{26P_2^2P_4 + 875P_3P_5}{a}\right) - 25P_4^2 \equiv 0 \text{ (mod. } a) = aS, \text{ suppose.}$$

Then, by actual calculation, the residue of S is found to be

$$-262500b^4e + 612500b^3cd - 339080b^2c^3.$$

Two reductions have already been made in obtaining this form S of the 5th degree. A final combination of S with P_2P_6 and the form (a^3e^2), whose value was given in a former lecture, enables us to divide out once more by a and thus get the form (a^2cg) of the 4th degree.

It is the fact that P_2P_6 and (a^3e^2) have residues which are not the same to a numerical factor près which necessitates the long calculation above described.

No linear combination of P_2P_6 and (a^3e^2) with one another is divisible by a, and it is necessary to find a third form S a linear combination of which with both P_2P_6 and (a^3e^2) will divide by a.

There is, however, another way of arriving at the form (a^2cg) by using the eductive generator

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \ldots .$$

Starting with the Quasi-Catalecticant

$$(a^2ce) = 800a^2ce - 1000ab^2e - 875a^2d^2 + 2450abcd - 1344ac^3 - 35b^2c^2,$$

and operating on it with G, we have

$$\begin{aligned} G(a^2ce) = {} & 4(ac - b^2)(-2000abe + 2450acd - 70bc^2) \\ & + 5(ad - bc)(800a^2e + 2450abd - 4032ac^2 - 70b^2c) \\ & + 6(ae - bd)(-1750a^2d + 2450abc) \\ & + 7(af - be)(800a^2c - 1000ab^2). \end{aligned}$$

The terms of this expression contain the common numerical factor 10, which may be rejected; thus we have

$$G(a^2ce) = 10(a^3cf),$$

where

$$\begin{aligned} (a^3cf) = {} & 560a^3cf - 700a^2b^2f - 650a^3de - 290a^2bce + 1500ab^3e \\ & + 2275a^2bd^2 - 1036a^2c^2d - 3710ab^2cd + 1988abc^3 + 63b^3c^2. \end{aligned}$$

This form (a^3cf) is the first educt of (a^2ce), and is irreducible (but, being of the fifth degree, does not appear in our list, which contains no forms of higher degree than the fourth). Operating on it with G, we obtain the educt of (a^3cf), which is the second educt of (a^2ce). This second educt will be of the 6th degree (its leading term will be a^4cg), but is reducible to the 5th when combined with

$$(4ac - 5b^2)(a^2ce),$$

as we know from the general theorem concerning the reduction of second educts. We shall thus obtain a form (a^3cg), the reduced second educt of (a^2ce), of the 5th degree, and a final combination of (a^3cg) with one or both of the forms P_2P_6 and (a^3e^2) will enable us to divide once more by a and thus arrive at (a^2cg) of the 4th degree.

By either of these methods we obtain

$$\begin{aligned} (a^2cg) = {} & 1176a^2cg - 8085a^2df + 7040a^2e^2 - 1470ab^2g + 18963abcf \\ & - 16940abde - 27160ac^2e + 26460acd^2 - 9555b^3f \\ & + 28098b^2ce + 12740b^2d^2 - 52822bc^2d + 21560c^4; \end{aligned}$$

but the second way, besides being more direct, gives us at the same time the value of the irreducible form (a^3cf).

Every Pure Reciprocant is an Invariant of a Binary Quantic whose coefficients $A, B, C, D, \ldots.$ are functions of the original elements $a, b, c, d, \ldots.$ such that

$$\begin{aligned} VA &= 0, \\ VB &= A, \\ VC &= 2B, \\ VD &= 3C, \\ &\ldots\ldots \end{aligned}$$

and conversely, every Invariant of this Binary Quantic, or of a system of such Binary Quantics, is a Pure Reciprocant.

This is a particular case of the more general theorem, due to Mr. Hammond, that if Θ is the operator,

$$\phi_1(a)\,\partial_b + \phi_2(a, b)\,\partial_c + \phi_3(a, b, c)\,\partial_d + \ldots.,$$

where $\phi_1, \phi_2, \phi_3, \ldots.$ are arbitrary rational integral functions, and if

$$A, B, C, D, \ldots., \; A', B', C', D', \ldots., \; A'', B'', C'', \ldots.$$

be any rational integral functions of the original letters $a, b, c, \ldots.$ which satisfy the conditions

$$\begin{array}{lll} \Theta A = 0, & \Theta A' = 0, & \Theta A'' = 0, \\ \Theta B = A, & \Theta B' = A', & \Theta B'' = A'', \\ \Theta C = 2B, & \Theta C' = 2B', & \Theta C'' = 2B'', \\ \Theta D = 3C, & \Theta D' = 3C', & \Theta D'' = 3C'', \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}$$

then every invariant in respect to the elements

$$A, B, C, D, \ldots., \; A', B', C', D', \ldots., \; A'', B'', C'', D'', \ldots.$$

is a rational integral solution of the equation

$$\Theta = 0.$$

Obviously, every rational integral solution of $\Theta = 0$ is an invariant in the above elements, so that the converse of the proposition is true. For the only conditions imposed upon $A, A', A'', \ldots.$ are that they shall be rational integral functions of $a, b, c, d, \ldots.$ annihilated by Θ. Let

$$\Phi(A, B, C, D, \ldots., \; A', B', C', D', \ldots., \; A'', B'', C'', D'', \ldots.)$$

be any invariant in the large letters. We have to show that

$$\Theta\Phi = 0.$$

Now,

$$\Theta\Phi = \frac{d\Phi}{dA}\Theta A + \frac{d\Phi}{dB}\Theta B + \frac{d\Phi}{dC}\Theta C + \dots\dots$$
$$+ \frac{d\Phi}{dA'}\Theta A' + \frac{d\Phi}{dB'}\Theta B' + \frac{d\Phi}{dC'}\Theta C' + \dots\dots$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Hence, writing for ΘA, ΘB, ΘC, ..., their values given above, we have

$$\begin{aligned}\Theta\Phi &= (A\partial_B + 2B\partial_C + 3C\partial_D + \dots\dots)\Phi \\ &+ (A'\partial_{B'} + 2B'\partial_{C'} + 3C'\partial_{D'} + \dots\dots)\Phi \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &= 0 \text{ (since } \Phi \text{ is an invariant)};\end{aligned}$$

which proves the proposition.

Confining our attention to a single set of letters, the Binary Quantic

$$(A, B, C, \dots\dots J, K, L \between x, y)^n,$$

whose coefficients are formed from one another by the successive operation of Θ as above, may be called a Quasi-Covariant; and it will follow immediately from the Theory of Binary Forms that every Covariant of a Quasi-Covariant is itself a Quasi-Covariant, and that every Invariant of any Quasi-Covariant (or system of Quasi-Covariants) is an Invariant in respect to the letters A, B, C, ..., and therefore, by what precedes, a rational integral solution of $\Theta = 0$.

Writing the terms of

$$(A, B, C, \dots\dots J, K, L \between x, y)^n$$

in reverse order, we have

$$Ly^n + nKxy^{n-1} + \frac{n(n-1)}{1.2} Jxy^{n-2} + \dots\dots + Ax^n,$$

where

$$\Theta L = nK, \ \Theta K = (n-1)J, \dots\dots \Theta A = 0.$$

Thus the Quasi-Covariant may be written

$$Ly^n + \Theta Lxy^{n-1} + \frac{\Theta^2 L}{1.2}x^2y^{n-2} + \dots\dots + \frac{\Theta^n L}{1.2.3\dots\dots n}x^n = y^n\left(e^{\frac{x\Theta}{y}}\right)L,$$

where $\Theta^{n+1}L = 0$.

This is the general symbolic expression for a Quasi-Covariant. An example of a Quasi-Covariant has already been given in Lecture II (Vol. VIII, p. 205), where it was stated, and afterwards proved (p. 256), that the reciprocal of the n^{th} modified derivative could be put under the form

$$-t^{-n-3}\left(e^{-\frac{V}{t}}\right)a_n.$$

The numerator of this reciprocal expression, which may be called the reciprocal function, is

$$t^n\left(e^{-\frac{V}{t}}\right)a_n,$$

which is identical with the general expression

$$y^n\left(e^{\frac{x\Theta}{y}}\right)L,$$

if $x = -1,\ y = t,\ L = a_n$ and $\Theta = V$.

Hence every Invariant of the reciprocal function is a Pure Reciprocant.

This property of the reciprocal function was discovered independently by Mr. C. Leudesdorf, who published his results in the Proceedings of the London Mathematical Society (Vol. XVII, p. 208). Mr. Hammond's results were given in two letters to me dated January 15th and January 20th, 1886, and were briefly alluded to by him at a meeting of the London Mathematical Society. They are here published for the first time.

Recalling the form of the operator

$$\Theta = \phi_1(a)\,\partial_b + \phi_2(a,\,b)\,\partial_c + \phi_3(a,\,b,\,c)\,\partial_d + \ldots\ldots,$$

where $\phi_1,\ \phi_2,\ \phi_3,\ \ldots\ldots$ are rational integral functions, we can form a Quasi-Covariant of extent j by a finite number of successive operations on a single letter of that extent.

To fix the ideas, take the letter d of extent 3, and operate on it with Θ; then

$$\Theta d = \phi_3(a,\,b,\,c).$$

Since $\phi_1,\ \phi_2,\ \phi_3,\ \ldots\ldots$ are by definition rational integral functions, we can, by operating a finite number of times with Θ, remove first c and then b from $\phi_3(a,\,b,\,c)$, and thus obtain

$$\Theta^n d = \text{funct. } a,$$

where n denotes a finite number of operations. Since $\Theta a = 0$, we have

$$\Theta^{n+1} d = 0.$$

In this manner we form the Quasi-Covariant of the n^{th} order

$$y^n\left(e^{\frac{x\Theta}{y}}\right)d.$$

If $\phi_2,\ \phi_3,\ \phi_4,\ \ldots\ldots$ do not contain higher powers than the first of the last letter in each, the order of the above Quasi-Covariant will be the same as its extent. This is the case with the reciprocal function, which is a co-reciprocant (*i. e.* a Quasi-Covariant relative to V).

Ex. $$y^2\left(e^{\frac{x}{V}}\right)c = cy^2 + Vcxy + \frac{V^2c}{1\,.\,2}x^2 = cy^2 + 5abxy + 5a^3x^2.$$

The discriminant of this is the pure reciprocant

$$5a^2\left(ac - \frac{5b^2}{4}\right).$$

As an additional example, consider the pair of linear co-reciprocants

$$4a(4ac - 5b^2)x + (5ad - 7bc)y,$$
$$50a(a^2d - 3abc + 2b^3)x + (25abd - 32ac^2 + 5b^2c)y.$$

The resultant of this pair is

$$2a(125a^3d^2 - 750a^2bcd + 500ab^3d + 256a^2c^3 + 165ab^2c^2 - 300b^4c),$$

i. e. is the Quasi-Discriminant multiplied by $2a$.

LECTURE XX.

"Quintessenced into a finer substance."—*Drummond of Hawthornden.*

Before proceeding with the proper subject of this day's lecture, I should like to mention a geometrical theorem which has fallen in my way, and which, *inter alia*, gives an immediate proof of the existence of 27 straight lines on a general cubic surface. It is proved by means of a Lemma (itself of quasi-geometrical origin) which finds its principal application in an extension of Bring's or Tschirnhausen's method, and shows how any number of specified terms, reckoning from either end, can be taken away from any equation of a sufficiently high degree.*

Subjectively speaking, I was led to the Lemma by considering the question, closely connected with Differential Invariants, of the method of depriving a linear differential equation of several terms.

Let ϕ be a cubic and u a linear function in x, y, z, t, say

$$\phi = ax^3 + \ldots . + fx^2y + \ldots . ,$$
$$u = lx + my + nz + pt.$$

Then, if ψ is a scroll which contains all the straight lines on $\phi + \lambda u^3$, when the parameter λ has any arbitrary numerical value from $+\infty$ to $-\infty$, I prove that

$$\psi = \phi^2 A + \phi u^3 B + u^6 C,$$

* I recover all Hamilton's results contained in his Report to the British Association, 1836, "On Jerrard's Method," in a much more clear and concise manner, and make important additions to his theory.

where ψ is of the degree 15 in the variables x, y, z, t,

.... 6 in the coefficients (l, m, n, p) of u,

.... 11 .... $(a, \ldots\ldots)$ of ϕ.

Or, more briefly, in	x	l	a
ψ is of degree	15	6	11, and consequently
C	9	0	11.

The intersections of ϕ with ψ are its intersections with u^6 and with C, of which the intersections with the arbitrary plane u^6 are clearly foreign to the question, but the cubic ϕ and the $9^\circ C$ intersect in 27 straight lines, which are the 27 ridges on ϕ.

C is identical with the covariant found by Clebsch and given in Salmon's Geometry of Three Dimensions at the end of the chapter on Cubic Surfaces. It may with propriety be called the Clebschian.

By giving the parameter λ (which occurs in $\phi + \lambda u^3$) an infinitesimal variation, it is easily proved that

$$B = -2EC, \; A = E^2 C, \; E^3 C = 0,$$

where E is the operator $l^3 \partial_a + \ldots + 3l^2 m \partial_f + \ldots$, which may be simply and completely defined by its property of changing the general cubic ϕ into $(lx + my + nz + pt)^3$.

The equation $E^3 C = 0$ expresses a new property of the Clebschian: it shows that if a, f are the coefficients of x^3 and any other term in ϕ containing x^2, neither a^3 nor $a^2 f$ can occur in any one of the terms of C. Defining a principal term in ϕ as one which contains the cube of one of the variables, and a term adjacent to it as one which contains the square of the same variable, this is equivalent to saying that neither the cube of the coefficient of a principal term nor its square multiplied by the coefficient of any adjacent term can appear in any of the terms of C.

An interesting special case of the general theorem is when the arbitrary plane u is taken to be one of the planes of reference, say $u = x$. Then

$$l = 1, \; m = 0, \; n = 0, \; p = 0,$$

and the operator E becomes simply $\frac{d}{da}$. Thus we learn that

$$\phi^2 \frac{d^2 C}{da^2} - 2x^3 \phi \frac{dC}{da} + x^6 C$$

is a Scroll of the fifteenth order which contains all the Ridges on

$$\phi + \lambda x^3$$

for any arbitrary value of the parameter λ.

It also contains 6 times over the curve of intersection of $\phi = 0$ with $x = 0$.

I now propose to give the substance, with a brief commentary, of some very interesting letters I have recently received from Capt. MacMahon. I abstain from giving a proof of his results, as I am informed that he intends to do this himself at an early meeting of the London Mathematical Society.

Using V to signify the Reciprocant Annihilator and Ω the Annihilator of Invariants, we have studied the properties of

$$V\frac{d}{dx} - \frac{d}{dx}V$$

and those of

$$\Omega\frac{d}{dx} - \frac{d}{dx}\Omega.$$

These may be written in the form

$$\begin{vmatrix} V & \frac{d}{dx} \\ V & \frac{d}{dx} \end{vmatrix} \qquad \begin{vmatrix} \Omega & \frac{d}{dx} \\ \Omega & \frac{d}{dx} \end{vmatrix},$$

and may be called alternants to $V, \frac{d}{dx}$ and to $\Omega, \frac{d}{dx}$ respectively.

It has been shown in Lecture VII (see Vol. VIII, p. 238 of this Journal) that

$$V\frac{d}{dx} - \frac{d}{dx}V = 2(3i + w)a.$$

The corresponding formula is

$$\Omega\frac{d}{dx} - \frac{d}{dx}\Omega = 3i + 2w,$$

as may be seen by writing $\kappa = 0,\ \lambda = 3,\ \mu = 4,\ \nu = 5, \ldots$ in a more general formula given in Lecture V (p. 224).

Observe that operating with the alternant to $\Omega, \frac{d}{dx}$ is equivalent to multiplication by a number, and that operating with the alternant to $V, \frac{d}{dx}$ merely introduces a numerical multiple of a as a factor. No such property exists for the Alternant

$$V\Omega - \Omega V,$$

but one much more extraordinary.

MacMahon has found that this alternant, which he calls J, is a generator to a Reciprocant and a generator to an Invariant; *i. e.* it converts a Reciprocant into another Reciprocant, and an Invariant into another Invariant. As regards a Differential Invariant, which is at once an Invariant and a Reciprocant, it is an Annihilator. He shows, in fact, that

$$\Omega J - J\Omega = 0$$

and

$$VJ - JV = 0.$$

If, then, $\Omega R = 0$, it follows immediately that $\Omega(JR) = 0$; *i. e.* if R is an invariant, JR is so too. And in like manner, if

$$VR = 0, \quad V(JR) = 0,$$

i. e. if R is a reciprocant, so is JR.

Of course, if M is a Differential Invariant,

$$JM = V(\Omega M) - \Omega(VM) = 0.$$

Let me here give a caution which may be necessary: The fact that a form is annihilated by J is not sufficient to show that it is a Differential Invariant, though all Differential Invariants are necessarily annihilated by J. Forms exist which are subject to annihilation by

$$J = a^2\partial_c + 3ab\partial_d + \ldots,$$

but are, notwithstanding, *neither* invariants nor reciprocants.

Such a form is the monomial b, which is obviously annihilated by J. Another is $ad - 3bc$. For, since

$$a^2d - 3abc + 2b^3$$

is a Differential Invariant, we have

$$J(a^2d - 3abc + 2b^3) = 0.$$

But

$$Jb^3 = 0 \quad \text{and} \quad Ja = 0;$$

therefore, also,

$$aJ(ad - 3bc) = 0.$$

The general theorem is as follows, and is a most remarkable one: If we write

$$\begin{aligned} mP(m, \mu, v, n) = \mu a^m \partial_{a_n} &+ (\mu + v)\, m a^{m-1} b \partial_{a_{n+1}} \\ &+ (\mu + 2v)\left(m a^{m-1} c + \frac{m(m-1)}{2} a^{m-2} b^2\right)\partial_{a_{n+2}} \\ &+ (\mu + 3v)\left\{m a^{m-1} d + m(m-1) a^{m-2} bc \right. \\ &\left. + \frac{m(m-1)(m-2)}{6} a^{m-3} b^3\right\}\partial_{a_{n+3}} + \ldots, \end{aligned}$$

where the coefficients of the terms inside the brackets are the same as those of the corresponding terms in the expansion of $(a+b+c+\ldots.)^m$, and where a_n stands for the n^{th} letter of the series $a, b, c, d, \ldots.$, then Capt. MacMahon establishes that *the alternant of any two P's is another P.*

A question here suggests itself naturally: What would be the alternant of three or more P's? For instance, would the alternant

$$\begin{vmatrix} P_1 & P_2 & P_3 \\ P_1 & P_2 & P_3 \\ P_1 & P_2 & P_3 \end{vmatrix} = P_1P_2P_3 - P_1P_3P_2 + P_2P_3P_1 - P_2P_1P_3 + P_3P_1P_2 - P_3P_2P_1$$

be another P?*

Moreover, he obtains expressions for the parameters m, μ, v, n of the resulting P in terms of the parameters of its two components. He proves that if P_1, P_2 are the two components whose alternant is P, supposing

$$m_1, \mu_1, v_1, n_1 \text{ to be the parameters of } P_1,$$
$$m_2, \mu_2, v_2, n_2 \ldots\ldots\ldots\ldots\ldots\ldots P_2,$$

then the parameters m, μ, v, n of their resultant P are given by the equations

$$m = m_1 + m_2 - 1,$$
$$\mu = (m_1 + m_2 - 1)\left\{\frac{\mu_2}{m_2}(\mu_1 + n_2v_1) - \frac{\mu_1}{m_1}(\mu_2 + n_1v_2)\right\},$$
$$v = (n_2 - n_1)v_1v_2 - \frac{m_2 - 1}{m_1}\mu_1v_2 + \frac{m_1 - 1}{m_2}\mu_2v_1,$$
$$n = n_1 + n_2.$$

It will be seen that Ω and V are special forms of P. Thus,

$$\Omega = P(1, 1, 1, 1),$$
$$V = P(2, 4, 1, 1).$$

Now, if the second and third parameters are zero, every term of P vanishes, and MacMahon finds that in the following two cases the second and third parameters of the resultant above given vanish.

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* In my Multiple Algebra investigations, which I hope some day to resume, I have made important use of similar Alternants, which, it may be noticed, do not vanish when their elements are non-commutative. In this connection it is well worthy of observation that the P's (as indeed would be true of any operators linear in the differential inverses) obey the *associative* law.

It would be interesting to ascertain under what arithmetical conditions, if any, other than MacMahon's, *any* two linear operators of the same general form as his P's become commutative.

Perhaps it would also be worthy of inquiry whether the P theory might not admit of extension in some form to operators non-linear in the differential inverses, and whether to every such operator of degrees i and j in the letters and their differential inverses there is not correlated another in which i and j are interchanged.

(1). Supposing $\frac{\mu_1}{m_1 v_1}$ to be an integer, this takes place when the two component system of parameters are

$$m_1,\ \mu_1,\quad v_1,\quad n_1,$$
$$m_2,\ \mu_1 m_2,\ m_1 v_1,\ n_1 + \frac{\mu_1}{m_1 v_1}(m_2 - m_1).$$

(2). When they are

$$m_1,\ \mu_1,\quad v_1,\quad n_1,$$
$$m_2,\ n_1 m_2,\ m_1 - 1,\ \frac{\mu_1}{m_1 v_1}(m_2 - 1).$$

Now,
$$P(1,\ 1,\ 1,\ 1) = \Omega,$$
$$P(2,\ 4,\ 1,\ 1) = V,$$

and by the law of composition

$$J = \Omega V - V\Omega = P(2,\ 2,\ 1,\ 2).$$

Also, $\left.\begin{matrix}2,\ 2,\ 1,\ 2\\ 1,\ 1,\ 1,\ 1\end{matrix}\right\}$ will be found to come under the first case;

and $\left.\begin{matrix}2,\ 2,\ 1,\ 2\\ 2,\ 4,\ 1,\ 1\end{matrix}\right\}$ . . . . the second.

Hence,
$$\Omega J - J\Omega = 0 \text{ and } VJ - JV = 0.$$

The above theorem is one of extraordinary beauty, and must play an important part in the future of Algebra.

In another letter Capt. MacMahon calls my attention to the fact that the operator called by me Cayley's generator P, in Lecture IV of this course (*American Journal of Mathematics*, Vol. VIII, p. 221), is a particular case of one of a much more general character given by him in the Quarterly Mathematical Journal (Vol. XX, p. 362).

He also states that every pure reciprocant, when multiplied by the needful power of a, is an invariant of the binary quantic

$$\{2.(2n+1)!\}a^{n+1} - n\{1!(2n+1)!\}a^{n-1}bt$$
$$+ \frac{n(n-1)}{1.2}\{2!(2n)!\}\left\{a^{n-2}c + \frac{n-2}{2}a^{n-3}b^2\right\}t^2$$
$$- \frac{n(n-1)(n-2)}{1.2.3}\{3!(2n-1)!\}\left\{a^{n-3}d + (n-3)a^{n-4}bc + \frac{(n-3)(n-4)}{1.2.3}a^{n-5}b^3\right\}t^3$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which I have written in the non-homogeneous form.

But this expression is (to a numerical factor près) identical with the numerator of $\frac{d^{n+2}x}{dy^{n+2}}$ when $t, a, b, \ldots.$ are taken to be the modified differential derivatives $\frac{dy}{dx}, \frac{1}{2}\frac{d^2y}{dx^2}, \frac{1}{2.3}\frac{d^3y}{dx^3}, \ldots.$ See my note on Burman's law for the Inversion of the Independent Variable (Supplement to the Philosophical Magazine for December, 1854).

The property that its invariants are pure reciprocants has already been proved in the lectures.

LECTURE XXI.

I take blame to myself for not earlier communicating to the class the substance of a note of Mr. Hammond's under date of January 20th, 1886, in which he makes an interesting application of the theorem that any invariant of the form

$$y^n \left(e^{\frac{x}{y} V}\right) F(a, b, c, \ldots.),$$

in which the function F is subject to the condition

$$V^{n+1}F = 0,$$

or of any combination of such forms, is a pure reciprocant.

Forms such as the above, whose invariants are pure reciprocants, he calls *co-reciprocants*. It follows that any covariant of one or more co-reciprocants is itself a co-reciprocant, for any invariant of a covariant is an invariant.

Taking F to be a single letter b, c, d, he forms the functions

$$(1) \qquad by + 2a^2x,$$

$$(2) \qquad cy^2 + 5abxy + 5a^3x^2,$$

$$(3) \qquad dy^3 + 3(2ac + b^2)xy^2 + 21a^2bx^2y + 14a^4x^3,$$

in which

$$2a^2 = Vb,$$

$$5ab = Vc, \quad 5a^3 = \frac{V^2c}{1.2},$$

$$3(2ac + b^2) = Vd, \quad 21a^2b = \frac{V^2d}{1.2}, \quad 14a^4 = \frac{V^3d}{1.2.3}.$$

On writing $y=t$, $x=-1$, it will be observed that these three forms are the numerators of

$$\frac{1}{3!}\frac{d^3x}{dy^3},\ \frac{1}{4!}\frac{d^4x}{dy^4},\ \frac{1}{5!}\frac{d^5x}{dy^5}.$$

The Jacobian of (1) and (2) is

$$(4ac-5b^2)\,ay;$$

the coefficient of ay is the familiar pure reciprocant $4ac-5b^2$.

The Jacobian of (1) and (3) is the determinant

$$\begin{vmatrix} b & 2a^3 \\ dy^2+(4ac-5b^2)\,xy & (2ac+b^2)\,y^2 \end{vmatrix},$$

which is divisible by y, giving the quotient

(4) $$(2a^2d-2abc-b^3)\,y+2a^2(4ac-5b^2)\,x.$$

This is

$$y\left(e^{\frac{x}{y}V}\right)(2a^2d-2abc-b^3),$$

the terms involving $\dfrac{x^2}{y}$, $\dfrac{x^3}{y^3}$, ... vanishing identically.

Looking at $2a^2d-2abc-b^3$ as the anti-source to a Co-reciprocant,* we might at first sight expect that it would give rise to a co-reciprocant of the third order in x, y, whereas we see it is the anti-source of a linear co-reciprocant.

We have $\quad V(2a^2d-2abc-b^3)=2a^2(4ac-5b^2)$.

Combining this with

$$V(a^2d-3abc+2b^3)=0 \quad \text{(the well-known Mongian),}$$

and dividing by a, he obtains

$$V(5ad-7bc)=4a\,(4ac-5b^2).$$

Hence

(5) $$(5ad-7bc)\,y+4a\,(4ac-5b^2)\,x$$

is a co-reciprocant. It is in fact (4) reduced in degree.

The Jacobian of (5) and of $cy^2+5abxy+5a^3x^2$, *i. e.*

$$\begin{vmatrix} 5ad-7bc & 4a\,(4ac-5b^2) \\ 2cy+5abx & 5aby+10a^3x \end{vmatrix},$$

* What differentiates Reciprocants from Invariants is that we have no reverser to V as O is to Ω in the theory of Invariants, *i. e.* no reverser which does not introduce an additional letter.

The coefficients of a covariant are obtained either from the source by continually operating with O, or from the anti-source by continually operating with Ω. But in the case of a co-reciprocant, we are only able to proceed in one direction (viz. from the anti-source, or coefficient of the highest power of y, to the source), as we have only one operator, V, at our disposal.

will divide by a, and gives the new linear co-reciprocant

(6) $$(25abd - 32ac^2 + 5b^2c)\,y + 50a\,(a^2d - 3abc + 2b^3)\,x.$$

The coefficient of y is of weight 4, but instead of giving rise to a co-reciprocant of the 4th order, we see that this again is the anti-source of a linear co-reciprocant.

The resultant of the two linear co-reciprocants (4) and (6) divided by a numerical multiple of a gives the well-known Quasi-Discriminant $125a^3d^2 + \ldots$, as was stated at the end of Lecture XIX.

The noticeable fact is that (including $by + 2a^2x$) there exist 3 linear independent co-reciprocants of extent 3. Probably there are no more, but this requires proof.

The promised land of Differential Invariants or Projective Reciprocants is now in sight, and the remainder of the course will be devoted to its elucidation. Twenty lectures have been given on the underlying matter, and probably ten more, at least, will have to be expended on this higher portion of the theory.

One is surprised to reflect on the change which has come over the face of Algebra in the last quarter of a century. It is now possible to enlarge to an almost unlimited extent on any branch of it. These thirty lectures, embracing only a fragment of the theory of reciprocants, might be compared to an unfinished epic in thirty cantos. Does it not seem as if Algebra had attained to the character of a fine art, in which the workman has a free hand to develop his conceptions as in a musical theme or a subject for painting? Formerly it consisted almost exclusively of detached theorems, but now-a-days it has reached a point in which every properly developed algebraical composition, like a skilful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas.

It is quite conceivable that the results we have been investigating may be descended upon from a higher and more general point of view. Many circumstances point to such a consummation being probable. But man must creep before he can walk or run, and a house cannot be built downwards from the roof. I think the mere fact that our work enables us to simplify and extend the results obtained by so splendid a genius as M. Halphen, is sufficient to convey to us the assurance that we have not been beating the wind or chasing a phantom, but doing solid work. Let me instance one single point: M. Halphen has

succeeded, by a prodigious effort of ingenuity, in obtaining the differential equation to a cubic curve with a given absolute invariant. His method involves the integration of a complicated differential equation. In the method which I employ the same result is obtained by a simple act of substitution in an exceedingly simple special form of Aronhold's S and T, capable of being executed in the course of a few minutes on half a sheet of paper, without performing any integration whatever. This will be seen to be a simple inference from the theorem invoked under three names, to which allusion has been made in a preceding lecture and the demonstration of which will shortly occupy our attention.

Before entering upon the theory of Differential Invariants, I think it desirable to bring forward the exceedingly valuable and interesting communication with which I have been favored by M. Halphen establishing *à priori* the existence of *invariants* in general.

SUR L'EXISTENCE DES INVARIANTS.

(*Extracted from a Letter of M. Halphen to Professor Sylvester.*)

Dans des théories diverses on a rencontré des Invariants sans qu'on ait pénétré la cause générale de leur existence. C'est cette lacune qu'il s'agit ici de faire disparaître.

1. Soient $A, B, \ldots . L$ des quantités auxquelles on puisse attribuer des valeurs *ad libitum*.

Une *substitution* consiste à remplacer ces quantités $(A, B, \ldots . L)$ par d'autres $(a, b, \ldots . l)$.

Les substitutions, que l'on doit considérer ici, sont *définies* par des relations algébriques, de forme supposée donnée, mais contenant des *paramètres arbitraires* $p, q, \ldots .$

$$(1) \qquad \begin{cases} a = f(A, B, \ldots . L;\ p, q, \ldots .), \\ b = f_1(A, B, \ldots . L;\ p, q, \ldots .), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Soit maintenant une seconde substitution, de même espèce, mais avec d'autres paramètres $\varpi, \chi, \ldots .$, et donnant lieu à $(\alpha, \beta, \ldots . \lambda)$, en sorte qu'on ait

$$(1 \text{ bis}) \qquad \begin{cases} \alpha = f(A, B, \ldots . L;\ \varpi, \chi, \ldots .), \\ \beta = f_1(A, B, \ldots . L;\ \varpi, \chi, \ldots .), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

2. DÉFINITION. Les substitutions dont il s'agit forment un GROUPE, si, quels que soient les paramètres $p, q, \ldots\ldots, \pi, \chi, \ldots\ldots$, ainsi que $A, B, \ldots\ldots L$, il existe des quantités $P, Q, \ldots\ldots$ vérifiant les égalités semblables

$$(1 \text{ ter}) \qquad \begin{cases} \alpha = f(a, b, \ldots\ldots l;\ P, Q, \ldots\ldots), \\ \beta = f_1(a, b, \ldots\ldots l;\ P, Q, \ldots\ldots), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Les invariants sont l'apanage exclusif des substitutions formant groupe. On va le montrer. Mais auparavant, pour éviter toute confusion, on doit faire une remarque sur la définition.

3. Dans les diverse théories où l'on a rencontré des Invariants, les substitutions forment groupe, en effet, suivant cette definition; mais il s'y rencontre encore une circonstance particulière de plus, c'est que les paramètres $P, Q, \ldots\ldots$ de la substitution composée (1 ter) dépendent uniquement des paramètres $p, q, \ldots\ldots, \pi, \chi, \ldots\ldots$ des substitutions composantes, (1) et (1 bis). Cette propriété *n'est pas nécessaire* à l'existence des Invariants, et nous ne la supposerons pas ici. Il sera donc entendu que $P, Q, \ldots\ldots$ peuvent dépendre, non seulement de $p, q, \ldots\ldots, \pi, \chi, \ldots\ldots$, mais aussi de $A, B, \ldots\ldots L$.

EXEMPLES :

$$\begin{array}{llll} \text{I.} & a = Ap^2, & b = Apq + Bp, & c = Aq^2 + 2Bq + C; \\ & \alpha = A\pi^2, & \beta = A\pi\chi + B\pi, & \gamma = A\chi^2 + 2B\chi + C; \\ & \alpha = aP^2, & \beta = aPQ + bP, & \gamma = aQ^2 + 2BQ + C; \end{array}$$

$$P = \frac{\pi}{p}, \qquad Q = \frac{\chi - q}{p}.$$

P et Q ne dépendent pas de A, B, C.

$$\begin{array}{llll} \text{II.} & a = A^3p^3, & b = A^2pq + ABp, & c = Aq^2 + 2Bq + C; \\ & \alpha = A^3\pi^3, & \beta = A^2\pi\chi + AB\pi, & \gamma = A\chi^2 + 2B\chi + C; \\ & \alpha = a^3P^3, & \beta = a^2PQ + abP, & \gamma = aQ^2 + 2bQ + c; \end{array}$$

$$P = \frac{\pi}{A^3p^3}, \qquad Q = \frac{\chi - q}{Ap}.$$

P et Q dépendent de A.

Dans ces deux exemples, il y a un invariant absolu, $\dfrac{B^2 - AC}{A}$.

4. Dans la substitution (1) nous supposerons que le nombre des paramètres soit inférieur au nombre des quantités $A, B, \ldots\ldots L$.

Soient ainsi m le nombre des paramètres $p, q, \ldots.$,
n le nombre des quantités $A, B, \ldots. L$,

on suppose $$m < n.$$

Cela étant, on peut éliminer les paramètres entre les équations (1), et il reste $(n - m)$ équations

$$(2) \qquad \begin{cases} F\,(a, b, \ldots. l;\ A, B, \ldots. L) = 0, \\ F_1(a, b, \ldots. l;\ A, B, \ldots. L) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

THÉORÈME: *Si les substitutions considérées forment* GROUPE, *les* $(n - m)$ *équations* (2) *peuvent être mises sous la forme*

$$(3) \qquad \begin{cases} \Phi\,(a, b, \ldots. l) = \Phi\,(A, B, \ldots. L), \\ \Phi_1(a, b, \ldots. l) = \Phi_1(A, B, \ldots. L), \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

en d'autres termes, *il y a* $(n - m)$ *invariants absolus.*

Réciproquement, s'il y a $(n - m)$ *invariants absolus* (distincts), *les substitutions forment groupe.*

5. DÉMONSTRATION. Prouvons d'abord la seconde partie, ou réciproque. Voici l'hypothèse: des équations (1), par élimination de $p, q, \ldots.$ resultent les équations (3).

Par conséquent, $A, B, \ldots. L$ et $a, b, \ldots. l$ étant quelconques, mais satisfaisant aux équations (3), on peut déterminer p, q, au moyen des équations (1).

Soient $A, B, \ldots. L, p, q, \ldots., \pi, \chi, \ldots.$ pris arbitrairement, et $a, b, \ldots. l, \alpha, \beta, \ldots. \lambda$ déterminés par (1) et (1 bis). Suivant l'hypothèse, on a

$$\Phi(a, b, \ldots. l) = \Phi(A, B, \ldots. L) \text{ et } \Phi(\alpha, \beta, \ldots. \lambda) = \Phi(A, B, \ldots. L);$$

donc $$\Phi(a, b, \ldots. l) = \Phi(\alpha, \beta, \ldots. \lambda), \text{ etc.}$$

Donc on peut déterminer $P, Q, \ldots.$ par les équations (1 ter), ce qu'il fallait démontrer.

Démontrons maintenant la première partie, ou théorème direct. Par hypothèse, $A, B, \ldots. L, p, q, \ldots., \pi, \chi, \ldots.$ étant pris à volonté et $a, b, \ldots. l, \alpha, \beta, \ldots. \lambda$ déterminés au moyen de (1) et (1 bis), il en résulte les relations (1 ter).

Des équations (1) résulte le système (2); de même, de (1 bis) et de (1 ter) resultent

$$(2\text{ bis})\quad \begin{cases} F\,(\alpha,\beta,\ldots.\lambda;\ A,B,\ldots.L)=0,\\ F_1(\alpha,\beta,\ldots.\lambda;\ A,B,\ldots.L)=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

$$(2\text{ ter})\quad \begin{cases} F\,(\alpha,\beta,\ldots.\lambda;\ a,b,\ldots.l)\ =0,\\ F_1(\alpha,\beta,\ldots.\lambda;\ a,b,\ldots.l)\ =0,\\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Je dis que le système (2 ter) résulte de (2) et de (2 bis).

En effet, $a, b, \ldots. l$ et $\alpha, \beta, \ldots. \lambda$ n'étant definis que par (1) et (1 bis), le système (2 ter) résulte de (1) et de (1 bis) par l'élimination de $p, q, \ldots.,$ $\pi, \chi, \ldots.$ et $A, B, \ldots. L$. Mais l'élimination de $p, q, \ldots.$ remplace le système (1) par le système (2), celle de $\pi, \chi, \ldots.$ remplace le système (1 bis) par (2 bis); donc (2 ter) résulte de l'élimination de $A, B, \ldots. L$ entre (2) et (2 bis).

Le système (2), (2 bis) est formé par $2(n-m)$ équations, et cependant l'élimination de n lettres $A, B, \ldots. L$, au lieu de donner $(n-2m)$ équations, en donne $(n-m)$, les équations (2 ter). Si donc on élimine seulement $(n-m)$ lettres $A, B, \ldots. G$, les m autres $H, \ldots. L$ disparaîtront d'elles-mêmes. Tirons $A, B, \ldots. G$ des équations (2), et nous aurons

$$\begin{aligned} A &= \Psi\,(a, b, \ldots. l;\ H, \ldots. L),\\ B &= \Psi_1(a, b, \ldots. l;\ H, \ldots. L),\\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Tirons de même $A, B, \ldots. G$ des équations (2 bis), et nous aurons

$$\begin{aligned} A &= \Psi\,(\alpha, \beta, \ldots. \lambda;\ H, \ldots. L),\\ B &= \Psi_1(\alpha, \beta, \ldots. \lambda;\ H, \ldots. L),\\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Le résultat de l'élimination est donc représenté par $(n-m)$ équations telles que

$$(4)\quad \begin{cases} \Psi\,(a, b, \ldots. l;\ H, \ldots. L) = \Psi\,(\alpha, \beta, \ldots. \lambda;\ H, \ldots. L),\\ \Psi_1(a, b, \ldots. l;\ H, \ldots. L) = \Psi_1(\alpha, \beta, \ldots. \lambda;\ H, \ldots. L),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

et l'on sait que $H, \ldots. L$ disparaissent, d'eux-mêmes, de ces équations.

En assignant donc à $H, \ldots. L$ des valeurs numériques à volonté, on voit donc bien que les équations résultants, équivalentes à (2 ter), ont la forme

$$\begin{aligned} \Phi\,(a, b, \ldots. l) &= \Phi\,(\alpha, \beta, \ldots. \lambda),\\ \Phi_1(a, b, \ldots. l) &= \Phi_1(\alpha, \beta, \ldots. \lambda),\\ &\ldots\ldots\ldots\ldots \end{aligned}$$

C'est ce qu'il fallait démontrer.

6. REMARQUES. Si les équations (4) sont rationnelles, la disparition de $H, \ldots. L$ exige que Ψ ait la forme suivante

$$\Psi = \Phi(a, b, \ldots. l)\,\Theta(H, \ldots. L) + \theta(H, \ldots. L),$$

et de même pour Ψ_1, etc. Sous cette forme, on voit que Θ et θ disparaissent dans les équations (4), et l'invariant résultant est Φ.

Mais, si les équations (4) sont irrationnelles, la disparition de $H, \ldots. L$ peut n'être pas immédiate. En assignant à $H, \ldots. L$ des valeurs numériques à volonté, comme on l'a dit dans la démonstration, c'est-à-dire en considérant $H, \ldots. L$ comme des *constantes arbitraires*, on voit les invariants se présenter avec des constantes arbitraires. Ceci ne doit pas étonner, puisqu'il s'agit ici d'invariants *absolus*, que l'on peut effectivement modifier en leur ajoutant des constantes arbitraires ou en les multipliant par des constantes arbitraires, sans troubler la propriété d'invariance.

L'analyse employée dans la démonstration fournit un moyen régulier de former les invariants; ce moyen consiste à éliminer les paramètres dans les équations (1), puis à résoudre par rapport à $(n-m)$ quantités $A, B, \ldots. G$. Mais, les substitutions forment groupe, on peut aussi résoudre par rapport à $a, b, \ldots. g$, en éliminant les paramètres.

EXEMPLE: $\quad a = Ap^2,\ b = Apq + Bp,\ c = Aq^2 + 2Bq + C.$

En résolvant par rapport à c, c'est-à-dire en tirant p, q des deux premières, on obtient

$$c = A\left(\frac{b-Bp}{Ap}\right)^2 + 2B\frac{b-Bp}{Ap} + C = \frac{b^2}{Ap^2} + C - \frac{B^2}{A} = \frac{b^2}{a} + C - \frac{B^2}{A}.$$

Voici l'invariant $C - \dfrac{B^2}{A}$.

En résolvant par rapport à b, on trouve $b = \sqrt{a}\sqrt{\dfrac{B^2-AC}{A} + c}$, ce qui donne l'invariant $\dfrac{B^2-AC}{A} + c$, où c est une constante arbitraire.

LECTURE XXII.

E pur si muove.

The theory still moves on. We have now emerged from the narrows and are entering on the mid-ocean of Differential Invariants, or of Principiants, as I have called them. These, it will now be seen, are perfectly defined by their property of being at one and the same time invariants and pure reciprocants. In other words, if P be a Principiant, it has both Ω and V for its annihilators. Thus, *ex. gr.*, the Mongian

$$A = a^2d - 3abc + 2b^3$$

is necessarily a Principiant. For

$$\Omega A = (a\partial_b + 2b\partial_c + 3c\partial_d)(a^2d - 3abc + 2b^3) = 0,$$

and at the same time

$$VA = \{2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d\}(a^2d - 3abc + 2b^3) = 0.$$

Among Pure Reciprocants, those only are entitled to rank as Principiants whose form is persistent (merely taking up an extraneous factor, but otherwise unchanged) under the most general homographic substitution (see Lecture XIII, *American Journal of Mathematics*, Vol. IX, p. 17). We have therefore to show that such reciprocants and no others are subject to annihilation by Ω.

With this end in view, let us consider the effect of substituting $\frac{x}{1+hx}$ for x and $\frac{y}{1+hx}$ for y in any rational integral function of y and its derivatives with respect to x. Suppose that, in consequence of this substitution, the function

$$F(y, y_1, y_2, y_3, \ldots y_n)$$

becomes changed into

$$F_1(x, y, y_1, y_2, y_3, \ldots y_n);$$

then the transformed function will be

$$F(Y, Y_1, Y_2, Y_3, \ldots Y_n),$$

where $X = \frac{x}{1+hx}$, $Y = \frac{y}{1+hx}$, and $Y_1, Y_2, Y_3, \ldots Y_n$ are the successive derivatives of Y with respect to X.

If, for the moment, we agree to consider h as an infinitesimal (we shall

afterwards give it a finite value), neglecting squares and higher powers of h, we may write

$$X = x - hx^2,$$
$$Y = y - hxy.$$

Hence, by n successive differentiations of Y with respect to X, neglecting squares of h whenever they occur, we deduce

$$\begin{aligned}
Y_1 &= y_1 + hxy_1 - hy, \\
Y_2 &= y_2 + 3hxy_2, \\
Y_3 &= y_3 + 5hxy_3 + 3hy_2, \\
Y_4 &= y_4 + 7hxy_4 + 8hy_3, \\
Y_5 &= y_5 + 9hxy_5 + 15hy_4, \\
&\cdots\cdots\cdots\cdots\cdots \\
Y_{n-1} &= y_{n-1} + (2n-3)\,hxy_{n-1} + (n-1)(n-3)\,hy_{n-2}, \\
Y_n &= y_n + (2n-1)\,hxy_n + n(n-2)\,hy_{n-1}.
\end{aligned}$$

The last of these, for instance, is obtained as follows:

We have $\quad Y_n = \dfrac{dY_{n-1}}{dX}.$

But
$$\frac{d}{dX} = \frac{1}{1-2hx}\cdot\frac{d}{dx} = (1+2hx)\frac{d}{dx},$$
and
$$\begin{aligned}
\frac{dY_{n-1}}{dx} &= \frac{d}{dx}\{y_{n-1} + (2n-3)\,hxy_{n-1} + (n-1)(n-3)\,hy_{n-2}\} \\
&= y_n + (2n-3)\,hxy_n + n(n-2)\,hy_{n-1}.
\end{aligned}$$
Consequently,
$$\begin{aligned}
Y_n &= (1+2hx)\frac{dY_{n-1}}{dx} \\
&= (1+2hx)\{y_n + (2n-3)\,hxy_n + n(n-2)\,hy_{n-1}\} \\
&= y_n + (2n-1)\,hxy_n + n(n-2)\,hy_{n-1}.
\end{aligned}$$

On substituting the above values of $Y, Y_1, Y_2, \ldots\ldots Y_n$ in the transformed function, we find immediately

$$F(Y, Y_1, Y_2, \ldots\ldots Y_n) = (1 + hx\nu + h\Theta)\,F(y, y_1, y_2, \ldots\ldots y_n),$$

where ν and Θ are the partial differential operators

$$\begin{aligned}
\nu &= -y\partial_y + y_1\partial_{y_1} + 3y_2\partial_{y_2} + 5y_3\partial_{y_3} + 7y_4\partial_{y_4} + \ldots\ldots, \\
\Theta &= -y\partial_{y_1} + 3y_2\partial_{y_3} + 8y_3\partial_{y_4} + 15y_4\partial_{y_5} + \ldots\ldots + n(n-2)\,y_{n-1}\partial_{y_n}.
\end{aligned}$$

Changing to our usual notation, we write

$$y_1 = t,\ y_2 = 2a,\ y_3 = 2.3b,\ y_4 = 2.3.4c,\ \ldots\ldots,$$

and then if F_1 is what F (a rational integral function of $a, b, c, \ldots.$) becomes when we substitute $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ for x, y (regarding h as *infinitesimal*), we have

$$F_1 = (1 + hx\nu + h\Theta)\,F,$$

where $\qquad \nu = -y\partial_y + t\partial_t + 3a\partial_a + 5b\partial_b + 7c\partial_c + 9d\partial_d + \ldots.,$

and $\qquad \Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots.$

In general ν is merely the partial differential operator written above; but when its subject, F, is homogeneous, of degree i, and isobaric, of weight w, in

the letters	y,	t,	a,	b,	c,	d, ...	supposed to be
of degrees	1,	1,	1,	1,	1,	1, ...	
and of weights	-2,	-1,	0,	1,	2,	3, ...,	

its operation is equivalent to multiplication by the number $3i + 2w$. For in this case we have

$$y\partial_y + t\partial_t + a\partial_a + b\partial_b + c\partial_c + d\partial_d + \ldots. = i,$$

and $\qquad -2y\partial_y - t\partial_t \qquad + b\partial_b + 2c\partial_c + 3d\partial_d + \ldots. = w;$

so that we may regard ν as a number, simply writing

$$\nu = 3i + 2w$$

when we have occasion to do so.

We are now able to show that if F is a persistent form, we must necessarily have

$$\Theta F = 0.$$

For

$$\frac{F_1}{F} = 1 + \nu hx + \frac{h\Theta F}{F};$$

and consequently, if F_1 is divisible by F (this is what is meant by saying that F is a persistent form), unless ΘF vanishes, $\frac{\Theta F}{F}$ must be a rational integral function of $y, t, a, b, c, \ldots.$ But since the operation of Θ diminishes the weight by unity without altering the degree, $\frac{\Theta F}{F}$ must be of degree 0 and weight -1. The impossibility of the existence of such a function leads to the necessary conclusion that

$$\Theta F = 0.$$

Let us apply this result to the case of a pure reciprocant. We have

$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots. = -y\partial_t + \Omega.$$

Thus when F is a pure reciprocant, or indeed any function in which t does not appear, $y\partial_t F = 0$ and Θ reduces to Ω. We have therefore shown, in what precedes, that the condition

$$\Omega F = 0$$

is necessary to ensure the persistence of the form of F under a particular homographic substitution; *à fortiori,* this condition is also necessarily satisfied when the form of F is persistent under the most general homographic substitution $\left(\text{in which } x,\ y \text{ are changed into } \dfrac{lx+my+n}{l''x+m''y+n''},\ \dfrac{l'x+m'y+n'}{l''x+m''y+n''}\right)$.

The satisfaction of $\Omega F=0$ is of itself inadequate to ensure persistence under the general homographic substitution; the necessary and sufficient condition of pure reciprocants

$$VF=0$$

must also be satisfied. This follows from the fact that the general linear substitution, for which all pure reciprocants are persistent, is merely a particular case of the most general homographic substitution.

It only remains to be proved that the two conditions $VF=0$, $\Omega F=0$, taken conjointly, are sufficient as well as necessary.

In what follows I use a method which may be termed that of composition of variations. Its nature and value will be better understood if I first apply it to the rigorous demonstration of the theorem that the substitution of $x+hy$ for x in the Quantic

$$(a,\ b,\ c,\ \ldots\ \between x,\ y)^n$$

changes any function whatever of its coefficients, say

$$F(a,\ b,\ c,\ \ldots),\ \text{into}\ e^{h\Omega}F(a,\ b,\ c,\ \ldots).$$

This is not proved, but only verified up to terms of the second order of differentiation, in Salmon's Modern Higher Algebra (3d ed. 1876, p. 59). Remembering that, whatever the order n of the Quantic may be, the changed values of the coefficients $a,\ b,\ c,\ d,\ \ldots$ are

$$\begin{aligned} a' &= a, \\ b' &= b+ah, \\ c' &= c+2bh+ah^2, \\ d' &= d+3ch+3bh^2+ah^3, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

what we have to prove is that, for all values of h,

$$F(a',\ b',\ c',\ d',\ \ldots)=e^{h\Omega}F(a,\ b,\ c,\ d,\ \ldots).$$

In other words, if for brevity we write

$$F(a,\ b,\ c,\ \ldots)=F,$$

and

$$F(a',\ b',\ c',\ \ldots)=F_1,$$

it is required to show that

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}\Omega^2 F + \frac{h^3}{1.2.3}\Omega^3 F + \ldots,$$

where $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$.

When h is infinitesimal, it is obvious that

$$F_1 = F + h\Omega F.$$

Hence, when h has a general value, we may assume

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}P + \frac{h^3}{1.2.3}Q + \frac{h^4}{1.2.3.4}R + \ldots$$

Let h be increased by the infinitesimal quantity ε; then, considering this increase as resulting from a second substitution similar to the first, we see that F_1 becomes $F_1 + \varepsilon\Omega F_1$.

But it also becomes

$$F + (h+\varepsilon)\Omega F + \frac{(h+\varepsilon)^2}{1.2}P + \frac{(h+\varepsilon)^3}{1.2.3}Q + \ldots = F_1 + \varepsilon\frac{dF_1}{dh}$$

$$= F_1 + \varepsilon\left(\Omega F + hP + \frac{h^2}{1.2}Q + \frac{h^3}{1.2.3}R + \ldots\right).$$

Equating this to $F_1 + \varepsilon\Omega F_1$, we obtain

$$\Omega F_1 = \Omega F + hP + \frac{h^2}{1.2}Q + \frac{h^3}{1.2.3}R + \ldots$$

But $$\Omega F_1 = \Omega\left(F + h\Omega F + \frac{h^2}{1.2}P + \frac{h^3}{1.2.3}Q + \ldots\right).$$

The comparison of these two expressions gives

$$P = \Omega^2 F,$$
$$Q = \Omega P = \Omega^3 F,$$
$$R = \Omega Q = \Omega^4 F.$$
$$\ldots\ldots\ldots\ldots$$

Substituting these values in the assumed expansion for F_1, there results

$$F_1 = F + h\Omega F + \frac{h^2}{1.2}\Omega^2 F + \frac{h^3}{1.2.3}\Omega^3 F + \ldots,$$

which is the expanded form of

$$F_1 = e^{h\Omega}F.$$

A similar method of procedure will enable us to establish the corresponding but more elaborate formula

$$F_1 = (1 + hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F,$$

in which F is any *homogeneous* and *isobaric* function* of degree i and weight w in y and its modified derivatives $(t, a, b, c, \ldots.)$ with respect to x; the operator $\Theta = -y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots.$; the function F_1 is what F becomes in consequence of the substitution of $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ for x, y; h is any finite quantity, and $\nu = 3i + 2w$.

Before giving the proof of this theorem, I will show that, upon the assumption of its truth, two inverse finite substitutions will, as they ought, nullify each other, leaving the function operated upon unaltered in form.

To avoid needless periphrasis, we call the substitution of $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ for x, y the substitution h.

Either of the two substitutions, h, $-h$, reverses the effect of the other; for the substitution $-h$ turns

$$\frac{x}{1+hx} \text{ into } \frac{x}{1-hx} \div 1 + \frac{hx}{1-hx} = x,$$

and

$$\frac{y}{1+hx} \text{ into } \frac{y}{1-hx} \div 1 + \frac{hx}{1-hx} = y.$$

The two substitutions h, $-h$, performed successively on F, ought therefore to leave its value unaltered. But by hypothesis the substitution h converts F into F_1; consequently the substitution $-h$ performed on F_1 ought to change it back again into F.

It must be carefully observed that (since the operation of Θ decreases the weight by unity, leaving the degree unchanged) the weight of $\Theta^\kappa F$ is κ units lower than that of F, whilst the degree is the same for both.

Thus for $\quad F$ we have $3i + 2w = \nu$

and for $\quad \Theta^\kappa F \quad 3i + 2(w - \kappa) = \nu - 2\kappa$.

Hence the substitution $-h$, which changes

$$F \text{ into } (1-hx)^\nu e^{-\frac{h\Theta}{1-hx}} F,$$

also changes

$$\Theta F \text{ `` } (1-hx)^{\nu-2} e^{-\frac{h\Theta}{1-hx}} \Theta F,$$

$$\Theta^2 F \text{ `` } (1-hx)^{\nu-4} e^{-\frac{h\Theta}{1-hx}} \Theta^2 F,$$

. . . .

and in general

$$\Theta^\kappa F \text{ into } (1-hx)^{\nu-2\kappa} e^{-\frac{h\Theta}{1-hx}} \Theta^\kappa F.$$

*F need not be integral or even rational; whenever it is homogeneous or isobaric, ν will be a number.

Moreover, $1+hx$ becomes $1+\dfrac{hx}{1-hx}=(1-hx)^{-1}$, so that

$$(1+hx)^{\nu-\kappa}\Theta^\kappa F \text{ becomes } (1-hx)^{-(\nu-\kappa)}(1-hx)^{\nu-2\kappa}e^{-\frac{h\Theta}{1-hx}}\Theta^\kappa F$$
$$=(1-hx)^{-\kappa}e^{-\frac{h\Theta}{1-hx}}\Theta^\kappa F$$
$$=e^{-\frac{h\Theta}{1-hx}}(1-hx)^{-\kappa}\Theta^\kappa F \text{ (since } \Theta \text{ does not act on } x).$$

Consequently,
$$(1+hx)^\nu F \quad \text{becomes } e^{-\frac{h\Theta}{1-hx}}F,$$
$$(1+hx)^{\nu-1}\Theta F \quad " \quad e^{-\frac{h\Theta}{1-hx}}(1-hx)^{-1}\Theta F,$$
$$(1+hx)^{\nu-2}\Theta^2 F \quad " \quad e^{-\frac{h\Theta}{1-hx}}(1-hx)^{-2}\Theta^2 F,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

And since, by the formula to be verified,

$$F_1=(1+hx)^\nu F+h(1+hx)^{\nu-1}\Theta F+\frac{h^2}{1.2}(1+hx)^{\nu-2}\Theta^2 F+\ldots,$$
$$F_1 \text{ becomes } e^{-\frac{h\Theta}{1-hx}}\left\{1+h(1-hx)^{-1}\Theta+\frac{h^2}{1.2}(1-hx)^{-2}\Theta^2+\ldots\right\}F$$
$$=e^{-\frac{h\Theta}{1-hx}}e^{\frac{h\Theta}{1-hx}}F=F.$$

LECTURE XXIII.

We now proceed to show how the composition of variations can be made to furnish a strict proof of the formula

$$F_1=(1+hx)^\nu e^{\frac{h\Theta}{1+hx}}F,$$

which was set forth in the preceding lecture.

As before, calling the change of x, y into $\dfrac{x}{1+hx}$, $\dfrac{y}{1+hx}$, the substitution h, it is easy to see that the *product* of two substitutions, h, ε, is the substitution $h+\varepsilon$. For

$$\frac{x}{1+hx}\div 1+\varepsilon\frac{x}{1+hx}=\frac{x}{1+(h+\varepsilon)x},$$
$$\frac{y}{1+hx}\div 1+\varepsilon\frac{x}{1+hx}=\frac{y}{1+(h+\varepsilon)x}.$$

This shows that if

	F_1 is what F becomes on making the substitution h,
and	F_2 " " F_1 " " " " " ε,
then	F_2 " " F " " " " " $h+\varepsilon$.

Thus we can find two expressions for F_2, the comparison of which will enable us to assign the coefficients of all the powers of h in the expanded values of F_1.

The first two terms of this expansion were obtained, in the preceding lecture, by treating h as an infinitesimal. We may therefore write

$$F_1 = F + h(\nu x + \Theta)F + \frac{h^2}{1.2}N_2 + \frac{h^3}{1.2.3}N_3 + \ldots .$$

Changing h into $h + \varepsilon$, we deduce

$$F_2 = F + (h+\varepsilon)(\nu x + \Theta)F + \frac{(h+\varepsilon)^2}{1.2}N_2 + \frac{(h+\varepsilon)^3}{1.2.3}N_3 + \ldots .$$

For greater simplicity, let ε be an infinitesimal, and write

$$\frac{F_2 - F_1}{\varepsilon} = \Delta F_1 .$$

Then $$\Delta F_1 = (\nu x + \Theta)F + hN_2 + \frac{h^2}{1.2}N_3 + \ldots .$$

Now look at each term in the expansion of F_1 and find its increment (*i. e.* its Δ) when x, y undergo the substitution ε. We thus obtain

$$\Delta F_1 = \Delta F + h\Delta(\nu x + \Theta)F + \frac{h^2}{1.2}\Delta N_2 + \frac{h^3}{1.2.3}\Delta N_3 + \ldots .$$

Comparing these two values of ΔF_1, we find

$$N_2 = \Delta(\nu x + \Theta)F,$$
$$N_3 = \Delta N_2,$$
$$N_4 = \Delta N_3,$$
$$\cdots\cdots\cdots$$

and generally $$N_r = \Delta N_{r-1}.$$

These equations are sufficient to determine all the coefficients of F_1; it only remains to show how the operations Δ may be performed.

We have in fact

$$F_1 = F + h\Delta F + \frac{h^2}{1.2}\Delta^2 F + \frac{h^3}{1.2.3}\Delta^3 F + \ldots ,$$

where $$\Delta F = (\nu x + \Theta)F.$$

But we must not from this rashly infer that

$$\Delta^n F = (\nu x + \Theta)^n F.$$

To do so would be tantamount to regarding ν as a constant number, whereas its value depends on the degree and weight of the subject of operation.

This will be clearly seen in the calculation which follows.* We first generalize the formula $$\Delta F = (\nu x + \Theta) F$$
by making $\Theta^\kappa F$ the operand instead of F.

Then, since i is the degree and $w - \kappa$ the weight of $\Theta^\kappa F$, instead of
$$3i + 2w = \nu,$$
we have
$$3i + 2(w - \kappa) = \nu - 2\kappa.$$
Thus,
$$\Delta\Theta^\kappa F = \{(\nu - 2\kappa) x + \Theta\} \Theta^\kappa F.$$
Again, since
$$\Delta x = \left(\frac{x}{1 + \varepsilon x} - x\right) \div \varepsilon = -x^2,$$
we find
$$\Delta x^\lambda \Theta^\kappa F = \lambda x^{\lambda - 1}\Theta^\kappa F . \Delta x + x^\lambda \Delta\Theta^\kappa F = -\lambda x^{\lambda + 1}\Theta^\kappa F + x^\lambda\{(\nu - 2\kappa) x + \Theta\} \Theta^\kappa F.$$
Hence we obtain the general formula
$$\Delta x^\lambda \Theta^\kappa F = x^\lambda\{(\nu - 2\kappa - \lambda) x + \Theta\} \Theta^\kappa F,$$
by means of which we calculate in succession the values of $\Delta^2 F$, $\Delta^3 F$, ...
Thus,
$$\begin{aligned}\Delta^2 F &= \Delta(\nu x + \Theta) F \\ &= \nu \Delta x F + \Delta\Theta F \\ &= \nu x\{(\nu - 1) x + \Theta\} F + \{(\nu - 2) x + \Theta\}\Theta F \\ &= \{\nu(\nu - 1) x^2 + 2(\nu - 1) x\Theta + \Theta^2\} F.\end{aligned}$$
Hence
$$\begin{aligned}\Delta^3 F &= \nu(\nu - 1)\Delta x^2 F + 2(\nu - 1)\Delta x\Theta F + \Delta\Theta^2 F \\ &= \nu(\nu - 1) x^2\{(\nu - 2) x + \Theta\} F + 2(\nu - 1) x\{(\nu - 3) x + \Theta\}\Theta F \\ &\quad + \{(\nu - 4) x + \Theta\}\Theta^2 F \\ &= \{\nu(\nu - 1)(\nu - 2) x^3 + 3(\nu - 1)(\nu - 2) x^2\Theta + 3(\nu - 2) x\Theta^2 + \Theta^3\} F.\end{aligned}$$

If $[\nu]^n$ is used to denote $\nu(\nu - 1)(\nu - 2) \ldots$ to n factors ($[\nu]^1$ will of course mean ν), we have shown that
$$\begin{aligned}\Delta F &= ([\nu]^1 x + \Theta) F, \\ \Delta^2 F &= ([\nu]^2 x^2 + 2[\nu - 1]^1 x\Theta + \Theta^2) F, \\ \Delta^3 F &= ([\nu]^3 x^3 + 3[\nu - 1]^2 x^2\Theta + 3[\nu - 1]^1 x\Theta^2 + \Theta^3) F,\end{aligned}$$

* If our sole object were to show that $\Theta F = 0$ is a sufficient as well as necessary condition of the persistence of F, we might dispense with all further calculation. Thus it is obvious that, since $\Delta F = (\nu x + \Theta) F$, $\Delta^n F$ must be of the form $(x, \Theta)^n F$; for the dependence of ν on the degree-weight of the operand will not affect the *form* of Δ^n, but only its numerical coefficients. Hence we conclude that F_1 is of the form $\phi(x, \Theta) F$; and remembering that $\Theta^2 F = 0$, $\Theta^3 F = 0$, ... whenever $\Theta F = 0$, it is at once seen that not only (as was shown in the last lecture) must ΘF vanish when F is persistent under the substitution h, but, conversely, that when $\Theta F = 0$, the altered value of F contains the original value as a factor (the other factor being in this case a function of x only); *i. e.* F is persistent.

and by induction it may be proved that in general

$$\Delta^n F = \left\{[\nu]^n x^n + n[\nu-1]^{n-1}x^{n-1}\Theta + \frac{n(n-1)}{1.2}[\nu-2]^{n-2}x^{n-2}\Theta^2 + \ldots + \Theta^n\right\}F.$$

That the last term of this expression is $\Theta^n F$ is sufficiently obvious; what we wish to prove is that, when m is any positive integer less than n, the term in $\Delta^n F$ which involves Θ^m will be

$$\frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu-m]^{n-m}x^{n-m}\Theta^m F.$$

To find the term involving Θ^m in $\Delta^{n+1}F$, we need only consider the operation of Δ on two consecutive terms of $\Delta^n F$; none of the remaining terms will affect the result. Suppose, then, that

$$\Delta^n F = \ldots + px^{n-m}\Theta^m F + qx^{n-m+1}\Theta^{m-1}F + \ldots$$

Operating with Δ, we find

$$\begin{aligned}\Delta^{n+1}F &= \ldots + p\Delta x^{n-m}\Theta^m F + q\Delta x^{n-m+1}\Theta^{m-1}F + \ldots \\ &= \ldots + px^{n-m}\{(\nu-n-m)x+\Theta\}\Theta^m F \\ &\qquad + qx^{n-m+1}\{(\nu-n-m+1)x+\Theta\}\Theta^{m-1}F + \ldots \\ &= \ldots + \{p(\nu-n-m)+q\}x^{n+1-m}\Theta^m F + \ldots\end{aligned}$$

Now, assuming the general term of $\Delta^n F$ to be as written above, we have

$$p = \frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu-m]^{n-m},$$

$$q = \frac{n(n-1)\ldots(n-m+2)}{1.2.3\ldots(m-1)}[\nu-m+1]^{n-m+1};$$

so that $$q = p\left\{\frac{m(\nu-m+1)}{n-m+1}\right\}.$$

Thus the general term of $\Delta^{n+1}F$ has for its numerical coefficient

$$\begin{aligned}p(\nu-n-m)+q &= p\left\{\frac{m(\nu-m+1)+(\nu-n-m)(n-m+1)}{n-m+1}\right\} \\ &= p\left\{\frac{(n+1)(\nu-n)}{n-m+1}\right\} = \frac{(n+1)n\ldots(n-m+2)}{1.2.3.m}[\nu-m]^{n+1-m},\end{aligned}$$

which shows that the numerical coefficients in $\Delta^{n+1}F$ obey the same law as those in $\Delta^n F$; and as this law is true for $n = 1, 2, 3$, it is also true universally.

We have thus shown that the general term in

$$\Delta^n F \text{ is } \frac{n(n-1)\ldots(n-m+1)}{1.2.3\ldots m}[\nu-m]^{n-m}x^{n-m}\Theta^m F,$$

and, consequently, the corresponding general term in

$$\frac{h^n\Delta^n F}{1.2.3\ldots.n} \text{ is } \frac{[\nu-m]^{n-m}}{1.2.3\ldots.(n-m)}h^{n-m}x^{n-m}.\frac{h^m\Theta^m F}{1.2.3\ldots.m}.$$

Now, as we have already seen,

$$F_1=\left(1+h\Delta+\frac{h^2}{1.2}\Delta^2+\frac{h^3}{1.2.3}\Delta^3+\ldots.\right)F,$$

which, by merely expressing the symbolic factor as a series of powers of Θ, may be transformed into

$$\begin{aligned}F_1=&\left(1+[\nu]'hx+\frac{[\nu]^2}{1.2}h^2x^2+\frac{[\nu]^3}{1.2.3}h^3x^3+\ldots.\right)F,\\&+\left(1+[\nu-1]'hx+\frac{[\nu-1]^2}{1.2}h^2x^2+\frac{[\nu-1]^3}{1.2.3}h^3x^3+\ldots.\right)h\Theta F\\&+\left(1+[\nu-2]'hx+\frac{[\nu-2]^2}{1.2}h^2x^2+\frac{[\nu-2]^3}{1.2.3}h^3x^3+\ldots.\right)\frac{h^2\Theta^2F}{1.2}\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

where, remembering that $[\nu]^n$ stands for $\nu(\nu-1)(\nu-2)\ldots.$ to n factors, it is evident that the functions of x which multiply F, $h\Theta F$, $\frac{h^2}{1.2}\Theta^2F$, ... are all of them binomial expansions. Hence we immediately obtain

$$\begin{aligned}F_1&=(1+hx)^\nu F+(1+hx)^{\nu-1}h\Theta F+(1+hx)^{\nu-2}\frac{h^2}{1.2}\Theta^2F+\ldots.\\&=(1+hx)^\nu\left\{1+(1+hx)^{-1}h\Theta+(1+hx)^{-2}\frac{h^2\Theta^2F}{1.2}+\ldots.\right\}F,\end{aligned}$$

and finally, $$F_1=(1+hx)^\nu e^{\frac{h\Theta}{1+hx}}F.$$

Mr. Hammond has remarked that, with a slight modification, the foregoing demonstration will serve to establish the analogous theorem, that

$$F_1=(1+ht)^{-\mu}e^{-\frac{hV_1}{1+ht}}F,$$

where, as before, F means any homogeneous and isobaric function of degree i and weight w in the letters $y, t, a, b, c, \ldots.$; and F_1 is what F becomes when, leaving y unaltered, we change x into $x+hy$, where h is any finite quantity. Instead of the operator

$$\Theta=-y\partial_t+a\partial_b+2b\partial_c+3c\partial_d+\ldots.=-y\partial_t+\Omega$$

we have $$-V_1=yt\partial_y+t^2\partial_t-2a^2\partial_b-5ab\partial_c-\ldots.=yt\partial_y+t^2\partial_t-V;*$$

* This theorem was stated without proof in Lecture VIII, where, through inadvertence, the term $yt\partial_y$ in the expression for V_1 was omitted.

and instead of $\nu = 3i + 2w$, a different number, $\mu = 3i + w$ (which I have called the characteristic), taken negatively.

If we suppose that

F_1 is what F becomes on changing x into $x + hy$,

and F_2 " " F " " " x " $x + \epsilon y$,

then F_3 " " F " " " x " $x + (h + \epsilon)y$.

Hence, if $$F_1 = F + hP + \frac{h^2}{1.2} Q + \frac{h^3}{1.2.3} R + \ldots,$$

we must have $$F_3 = F + (h + \epsilon) P + \frac{(h+\epsilon)^2}{1.2} Q + \frac{(h+\epsilon)^3}{1.2.3} R + \ldots$$
$$= F_1 + \epsilon \frac{dF_1}{dh} + \ldots$$

Thus, if ϵ be regarded as infinitesimal, and we write

$$\frac{F_3 - F_1}{\epsilon} = \Delta F_1,$$

it follows that $$\Delta F_1 = P + hQ + \frac{h^2}{1.2} R + \ldots$$

But, by the direct operation of Δ, we find

$$\Delta F_1 = \Delta F + h\Delta P + \frac{h^2}{1.2} \Delta Q + \ldots,$$

and, comparing these two values of ΔF_1,

$$P = \Delta F,$$
$$Q = \Delta P = \Delta^2 F,$$
$$R = \Delta Q = \Delta^3 F,$$
$$\ldots\ldots\ldots\ldots$$

Hence it follows that

$$F_1 = F + h\Delta F + \frac{h^2}{1.2} \Delta^2 F + \frac{h^3}{1.2.3} \Delta^3 F + \ldots$$

It remains to find the value of $\Delta^n F$. This can be effected by means of formulae given in Lecture VIII (*American Journal of Mathematics*, Vol. VIII, p. 245), where it is shown that

$$\Delta x = y,$$
$$\Delta y = 0,$$
$$\Delta t = -t^2,$$
$$\Delta a = -3at,$$
$$\Delta b = -4bt - 2a^2,$$
$$\Delta c = -5ct - 5ab,$$
$$\Delta d = -6dt - 6ac - 3b^2,$$
$$\Delta e = -7et - 7ad - 7bc,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

We now show that

$$\Delta F = -(\mu t + V_1) F,$$

where

$$V_1 = V - t^2\partial_t - yt\partial_y,$$

just as in the cognate theorem we had

$$\Delta F = (\nu x + \Theta) F.$$

Since F is a function of $y, t, a, b, c, \ldots$ without x, it is evident that

$$\begin{aligned} \Delta F &= \frac{dF}{dy}\Delta y + \frac{dF}{dt}\Delta t + \ldots \\ &= -t(t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots) F \\ &\quad -\{2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d + \ldots\} F, \end{aligned}$$

where the part of ΔF which is independent of t is $-VF$.

Now, $\quad y\partial_y + t\partial_t + a\partial_a + b\partial_b + c\partial_c + \ldots = i$

and $\quad -2y\partial_y - t\partial_t + b\partial_b + 2c\partial_c + \ldots = w;$

so that $\quad t\partial_t + 3a\partial_a + 4b\partial_b + 5c\partial_c + \ldots = 3i + w - y\partial_y - t\partial_t.$

Hence, writing $3i + w = \mu$,

$$\begin{aligned} \Delta F &= -t(\mu - y\partial_y - t\partial_t) F - VF \\ &= -(\mu t + V_1) F, \end{aligned}$$

where

$$V_1 = V - t^2\partial_t - yt\partial_y.$$

Observing that $V_1^\kappa F$ is of degree $i + \kappa$ and weight $w - \kappa$; since

$$3(i + \kappa) + (w - \kappa) = \mu + 2\kappa,$$

we see that

$$\Delta V_1^\kappa F = -\{(\mu + 2\kappa)t + V_1\} V_1^\kappa F.$$

Again,

$$\begin{aligned} \Delta t^\lambda V_1^\kappa F &= \lambda t^{\lambda-1} V_1^\kappa F . \Delta t + t^\lambda \Delta V_1^\kappa F \\ &= -\lambda t^{\lambda+1} V_1^\kappa F - t^\lambda\{(\mu + 2\kappa)t + V_1\} V_1^\kappa F. \end{aligned}$$

We thus obtain the formula

$$\Delta t^\lambda V_1^\kappa F = -t^\lambda\{(\mu + \lambda + 2\kappa)t + V_1\} V_1^\kappa F, \tag{1}$$

analogous to the one previously employed,

$$\Delta x^\lambda \Theta^\kappa F = x^\lambda\{(\nu - 2\kappa - \lambda)x + \Theta\}\Theta^\kappa F. \tag{2}$$

The remainder of the work will be step for step the same for this as for the previous theorem. In fact, by using (1) just as we used (2), we shall deduce

$$F_1 = (1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F, \tag{3}$$

just as we deduced the analogous formula

$$F_1 = (1 + hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F. \tag{4}$$

The reason of this is obvious: by interchanging x and t, μ and $-\nu$, Θ and $-V_1$, we interchange the formulae (1) and (2), (3) and (4).

It may be well to observe that if we use S_h to denote a substitution of such a nature that
$$S_\epsilon S_h = S_{h+\epsilon},$$
and if (regarding ϵ as an infinitesimal) we write
$$\frac{S_\epsilon - 1}{\epsilon} = \Delta,$$
then in general
$$S_h F = e^{h\Delta} F.$$
The proof of this proposition is virtually contained in what precedes.

LECTURE XXIV.

Whenever a rational integral function of $x, y, t, a, b, c, \ldots$ is persistent in form under the general linear substitution, it cannot contain explicitly either x, y or t, but must be a function of the remaining letters $a, b, c, \ldots$ (the successive modified derivatives, beginning with the second, of y with respect to x) alone.

For if, keeping y unaltered, we change x into $x + \alpha$, where α is any arbitrary constant which may be regarded as an infinitesimal, the derivatives $t, a, b, c, \ldots$ are not affected by this change, and consequently the function
$$F = F(x, y, t, a, b, c, \ldots) \text{ becomes } F + \alpha \frac{dF}{dx},$$
which cannot be divisible by F unless $\frac{dF}{dx} = 0$.

(The alternative hypothesis of $\frac{dF}{dx}$ being divisible by F is inadmissible, because F is a rational integral function.)

Hence F cannot contain x explicitly; and if we write $y + \beta$ for y, keeping x unchanged, we see, in like manner, that F cannot contain y explicitly.

Again, if in the function
$$F = F(t, a, b, c, \ldots)$$
we change x, y into $x + \alpha$, $y + \beta_{\prime} x + \beta$, the effect of this substitution will be to increase t by the arbitrary constant $\beta_{\prime}$, without altering any of the remaining derivatives $a, b, c, \ldots$

Hence, in order that the form of F may still be persistent, we must have $\frac{dF}{dt} = 0$; the reasoning being just the same as that by which $\frac{dF}{dx}$ was seen to vanish. Thus, F does not contain t explicitly. Moreover, the function

$$F = F(a, b, c, \ldots.)$$

must be both homogeneous and isobaric.

For the substitution of $\alpha_{\prime}x + \alpha$, $\beta_{\prime\prime}y + \beta_{\prime}x + \beta$ for x, y, respectively, will multiply the letters

$$a \quad , \quad b \quad , \quad c \quad , \quad d \quad , \ldots.$$

by

$$\beta_{\prime\prime}\alpha_{\prime}^{-2}, \quad \beta_{\prime\prime}\alpha_{\prime}^{-3}, \quad \beta_{\prime\prime}\alpha_{\prime}^{-4}, \quad \beta_{\prime\prime}\alpha_{\prime}^{-5}, \ldots.$$

Each term of F will therefore be multiplied by a positive power of $\beta_{\prime\prime}$ and a negative power of $\alpha_{\prime}$.

Let one of the terms of F be $a^{\lambda_0}b^{\lambda_1}c^{\lambda_2}d^{\lambda_3} \ldots.$ It will be multiplied by

$$\beta_{\prime\prime}^{\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots} \alpha_{\prime}^{-(2\lambda_0 + 3\lambda_1 + 4\lambda_2 + 5\lambda_3 + \ldots.)}.$$

In order that F may retain its form, this multiplier must be the same for every term of F, no matter what arbitrary values are assigned to $\alpha_{\prime}$ and $\beta_{\prime\prime}$. This can only happen when, for all terms of the function F, we have

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots. = \text{const.}$$

and

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots. = \text{const.},$$

i. e. when F is homogeneous and isobaric.

We have thus proved that among all the rational integral functions of x, y, t, a, b, c, ... the only ones persistent under the substitution of $\alpha + \alpha_{\prime}x$, $\beta + \beta_{\prime}x + \beta_{\prime\prime}y$ for x, y, respectively, are such as simultaneously satisfy the conditions of not explicitly containing x, y or t, and of being homogeneous and isobaric in the remaining letters a, b, c, ...

If F, any function satisfying these conditions, merely acquires an extraneous factor when, leaving y unaltered, we change x into $x + hy$, the form of F will be persistent under the general linear substitution. For both $\alpha + \alpha_{\prime}(x + hy)$ and $\beta + \beta_{\prime}(x + hy) + \beta_{\prime\prime}y$ are general linear functions of x, y, 1.

Now, the change of x into $x + hy$ converts (as was shown in the preceding lecture) F into

$$F_1 = (1 + ht)^{-\mu} e^{-\frac{hV_1}{1+ht}} F,$$

where

$$V_1 = V - t^2\partial_t - yt\partial_y.$$

But, since neither y nor t occurs in F, we must have

$$\partial_y F = 0 \text{ and } \partial_t F = 0.$$

Consequently,

$$V_1 F = VF, \quad V_1^2 F = V^2 F,$$

and so on. Hence

$$F_1 = (1+ht)^{-\mu} e^{-\frac{hV}{1+ht}} F$$
$$= (1+ht)^{-\mu} F - (1+ht)^{-\mu-1} hVF + (1+ht)^{-\mu-2} \frac{h^2V^2}{1.2} F - \ldots .$$

Unless VF, V^2F, V^3F, ... all of them vanish, F_1 cannot contain F as a factor. If it could, VF, V^2F, ... would all have to be divisible by F. But this is impossible; for VF, a rational integral function of $a, b, c, \ldots.$ whose weight is $w-1$, cannot be divisible by F, a rational integral function of weight w.

We must therefore have

$$VF = 0 \qquad \text{(which implies } V^2F = 0, \text{ etc.)}$$

as the necessary and sufficient condition of the persistence of the form of F under the general linear substitution. In other words, F must be a pure reciprocant.

In order that F may also be persistent in form under the general homographic substitution, it must (besides being a pure reciprocant) be subject to annihilation by the operator

$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots .$$

For it was seen, in the preceding lecture, that the special homographic substitution in which $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ are written instead of x, y, respectively, has the effect of changing any homogeneous and isobaric function F into F_1, where

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Theta}{1+hx}} F,$$
$$\Theta = \Omega - y\partial_t .$$

When the letter t does not occur in F, we may write $\partial_t F = 0$, so that Θ becomes simply Ω, and the above formula becomes

$$F_1 = (1+hx)^{\nu} e^{\frac{h\Omega}{1+hx}} F.$$

Hence it follows immediately that, when F is a rational integral function of the letters $a, b, c, \ldots.$, the condition $\Omega F = 0$ is sufficient as well as necessary to ensure the persistence of the form of F under the special homographic substitution we have employed.

But when F is a pure reciprocant it also satisfies the condition $VF = 0$, and it is the simultaneous satisfaction of $\Omega F = 0$ and $VF = 0$ that ensures the persistence of the form of F under the most general homographic substitution.

This may be shown by combining the substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ (for which F is persistent when, and only when, $\Omega F = 0$) with the general linear substitution (for which $VF = 0$ is the necessary and sufficient condition of the persistence of the form of F), so as to obtain the most general homographic substitution. Thus the linear substitution

$$\left.\begin{aligned} x &= lx_{,} + my_{,} + n \\ y &= l'x_{,} + m'y_{,} + n' \end{aligned}\right\},$$

when combined with

$$x_{,} = \frac{x_{,,}}{1+hx_{,,}}, \quad y_{,} = \frac{y_{,,}}{1+hx_{,,}},$$

gives the substitution

$$\left.\begin{aligned} x &= \frac{lx_{,,} + my_{,,} + n(1+hx_{,,})}{1+hx_{,,}} \\ y &= \frac{l'x_{,,} + m'y_{,,} + n'(1+hx_{,,})}{1+hx_{,,}} \end{aligned}\right\},$$

in which both the numerators are general linear functions.

By combining the substitution just obtained with the linear substitution

$$x_{,,} = \lambda x_{,,,} + \mu y_{,,,} + \nu, \quad y_{,,} = y_{,,,},$$

the denominator of each fraction is changed into a general linear function, and thus, by combining the special homographic substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$ with two linear substitutions, we arrive at the most general homographic substitution.

This proves that the necessary and sufficient condition of F being a *homographically persistent* form is the coexistence of the two conditions

$$VF = 0, \quad \Omega F = 0.$$

Thus a Projective Reciprocant, or Principiant, or Differential Invariant, combines the natures of a Pure Reciprocant and Invariant *in respect of the elements.*

Notice that every Pure Reciprocant is an Invariant of the Reciprocal Function $\left(i.\ e.\right.$ the numerator of the expression for $\frac{d^n x}{dy^n}$ in terms of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... , or what is the same in terms of the modified derivatives t, a, b, $\left.\ldots.\right)$, but the elements of such invariants are *not* the original simple elements, but more or less complicated functions of them.

What has just been stated is obvious from the fact that all invariants of the "reciprocal function" have been shown to be pure reciprocants (*vide* Lect. XIX).

The ordinary protomorph invariants of this function will have for their leading term a power of a multiplied by a single letter. Consequently, by reasoning previously employed in these lectures, every pure reciprocant will be a rational function of invariants of the Reciprocal Function divided by some power of a. Thus, for example, the Reciprocal Function

$$14a^4 - 21a^2bt + 3(2ac + b^2)t^2 - dt^3 = (\alpha, \beta, \gamma, \delta \between 1, -t)^3$$

if
$$\alpha = 14a^4,\ \beta = 7a^2b,\ \gamma = 2ac + b^2,\ \delta = d.$$

The two protomorph invariants of this reciprocal function are

$$\alpha\gamma - \beta^2 = 7a^4(4ac - 5b^2)$$

and
$$\alpha^2\delta - 3\alpha\beta\gamma + 2\beta^3 = 196a^6(a^2d - 3abc + 2b^3).$$

All other pure reciprocants of extent 3 may be rationally expressed in terms of a and the two protomorphs $4ac - 5b^2$, $a^2d - 3abc + 2b^3$; *i. e.* all pure reciprocants of extent 3 are invariants of the reciprocal function of extent 3.

The reasoning employed can be applied with equal facility to the general case of extent n.

Instead of $\frac{x}{1 + hx}$, $\frac{y}{1 + hx}$, let us consider the special homographic substitution $\frac{1}{x}$, $\frac{y}{x}$ employed by M. Halphen.

Writing
$$X = \frac{1}{x} \text{ and } Y = \frac{y}{x},$$

let $Y_1, Y_2, Y_3, \ldots.$ denote the successive derivatives of Y with respect to X, and $y_1, y_2, y_3, \ldots.$ those of y with respect to x. Then

$$\begin{aligned}
Y &= x^{-1}y,\\
Y_1 &= -x\left(y_1 - \frac{1}{x}y\right),\\
Y_2 &= x^3y_2,\\
Y_3 &= -x^5\left(y_3 + \frac{3}{x}y_2\right),\\
Y_4 &= x^7\left(y_4 + \frac{8}{x}y_3 + \frac{12}{x^2}y_2\right),\\
Y_5 &= -x^9\left(y_5 + \frac{15}{x}y_4 + \frac{60}{x^2}y_3 + \frac{60}{x^3}y_2\right),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Hence, if $a, b, c, d, \ldots.$ are the successive modified derivatives (beginning with

the second) of y with respect to x, and a', b', c', d', ... the corresponding modified derivatives of Y with respect to X, it follows immediately that

$$\begin{aligned}
a' &= \quad x^3 a, \\
b' &= -x^5\left(b + \frac{1}{x}a\right), \\
c' &= \quad x^7\left(c + \frac{2}{x}b + \frac{1}{x^2}a\right), \\
d' &= -x^9\left(d + \frac{3}{x}c + \frac{3}{x^2}b + \frac{1}{x^3}a\right), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Attributing the weights 0, 1, 2, 3, ... to the letters $a, b, c, d, \ldots.$, it is very easily seen that if F is any homogeneous and isobaric function of degree i and weight w,

$$F(a', b', c', \ldots.) = (-)^w x^{3i+2w} F\left(a, b + \frac{1}{x}a, c + \frac{2}{x}b + \frac{1}{x^2}a, \ldots.\right).$$

But we proved (in Lecture XXII) that for all values of h

$$F(a, b + ah, c + 2bh + ah^2, \ldots.) = e^{h\Omega} F(a, b, c, \ldots.).$$

Hence, making $h = \frac{1}{x}$, we obtain

$$F(a', b', c', d', \ldots.) = (-)^w x^{3i+2w} e^{\frac{\Omega}{x}} F(a, b, c, \ldots.),$$

which proves that the satisfaction of

$$\Omega F(a, b, c, \ldots.) = 0$$

is the necessary and sufficient condition for the persistence of the form of F under the Halphenian substitution $\frac{1}{x}, \frac{y}{x}$.

Similarly we might prove that $F(y, t, a, b, c, \ldots.)$, which contains y and t, but not x, is changed by the substitution $\frac{1}{x}, \frac{y}{x}$ into

$$(-)^w x^r e^{\frac{\Theta}{x}} F(y, t, a, b, c, \ldots.),$$

where
$$\Theta = -y\partial_t + a\partial_b + 2b\partial_c + \ldots. = \Omega - y\partial_t;$$

or we may deduce this result from the formula, demonstrated in the preceding lecture of this course,

$$F_1 = (1 + hx)^r e^{\frac{h\Theta}{1+hx}} F,$$

in which F_1 is what F becomes in consequence of the substitution $\frac{x}{1+hx}, \frac{y}{1+hx}$ impressed on the variables.

Let i be the degree and ω the weight measured by the sum of the orders of differentiation in each term of

$$F(y, t, a, b, c, \ldots.).$$

If we measure the weight by the sum of the orders of differentiation of every term of F diminished by 2 units for each letter in the term, then

$$w = \omega - 2i \text{ and } 2\omega - i = 3i + 2w = \nu.$$

Let $\qquad F(y, t, a, b, c, \ldots.)$ become $F'(y, t, a, b, c, \ldots.)$,
when we change

$$x \text{ into } qx + p \text{ and } y \text{ into } ry;$$

then $\qquad F'(y, t, a, b, c, \ldots.) = r^i q^{-w} F(y, t, a, b, c, \ldots.).$

A further substitution $\frac{x}{1+hx}$, $\frac{y}{1+hx}$, impressed on the variables in F', will convert the original variables into

$$\frac{qx}{1+hx} + p \text{ and } \frac{ry}{1+hx},$$

i. e. into $\qquad \dfrac{p(1+hx)+qx}{1+hx}$ and $\dfrac{ry}{1+hx}$.

The function F' is at the same time changed into

$$r^i q^{-w}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F(y, t, a, b, c, \ldots.).$$

If now, in the above, we write $p = h$, $q = -h^2$, $r = h$, we shall have changed the original variables x, y into $\frac{h}{1+hx}$, $\frac{hy}{1+hx}$, and the original function F into

$$h^i(-h^2)^{-w}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F = (-)^w h^{i-2w}(1+hx)^\nu e^{\frac{h\Theta}{1+hx}} F = (-)^w \left(\frac{1+hx}{h}\right)^\nu e^{\frac{h\Theta}{1+hx}} F.$$

Let h become infinite; then $\frac{h}{1+hx}$, $\frac{hy}{1+hx}$ and $(-)^w\left(\frac{1+hx}{h}\right)^\nu e^{\frac{h\Theta}{1+hx}} F$ become $\frac{1}{x}$, $\frac{y}{x}$ and $(-)^w x^\nu e^{\frac{\Theta}{x}} F$, showing that the substitution $\frac{1}{x}$, $\frac{y}{x}$ changes F into $(-)^w x^\nu e^{\frac{\Theta}{x}} F$.

(*To be continued.*)

A Memoir in the Theory of Numbers.

By Arthur S. Hathaway.

Part I.—*Introduction.*

1. It has seemed desirable to begin this memoir with a brief account of the labors of others in the same direction, in order that the object of the memoir and its relations to previous results may be more clearly understood.

The labors of Gauss, Kummer, Dirichlet, Kronecker, Dedekind, and others, have extended the scope of the theory of numbers far beyond its original limit of the science of the natural numbers $0, \pm 1, \pm 2, \pm 3, \ldots.$ Gauss began the extension by showing that the theory of quantities of the form $a + b\sqrt{-1}$, where a and b are natural numbers, is quite similar to the theory of natural numbers. Quantities of this form may be called *biquadratic* numbers or integers; and one biquadratic integer is divisible by another when the quotient is a biquadratic integer. The numbers ± 1, $\pm\sqrt{-1}$, like ± 1 in the theory of natural numbers, divide all biquadratic integers, and are therefore set apart as "units" or "improper divisors." Every biquadratic integer may be factored into a product of powers of non-factorable or "prime" biquadratic integers; and, disregarding possible changes of these primes by unit factors, the factoring can be accomplished in only one way. Biquadratic primes divide themselves into three classes:

(1). Natural prime numbers, as $3, 7, \ldots$ of the form $4n + 3$.

(2). The conjugate biquadratic factors of natural primes of the form $4n + 1$.

(3). The number $1 + \sqrt{-1}$.

Following the development of the theory of biquadratic numbers came a similar development of the theory of *cubic* numbers or quantities of the form $a + bt$, where a, b are natural numbers and t is a primitive cube root of unity, viz. $t^2 + t + 1 = 0$. The cubic primes are also divided into three classes:

(1). Natural primes, as $2, 5, \ldots$ of the form $3n + 2$.

(2). The conjugate cubic factors of natural primes of the form $3n + 1$.

(3). The numbers $1 - t$, $1 - t^2$.

It is worthy of remark, as illustrating the manner in which mathematical theories of importance often originate, that neither of the preceding theories was developed systematically and for its own sake, but that each was the result of an effort to solve the problems of *biquadratic* and *cubic reciprocity* in the theory of numbers.

Another system of quantities whose theory is well known to be similar to the theory of natural numbers is the system of rational entire functions of an indeterminate x.

2. An important feature of all these theories is the fact that Euclid's process of finding the greatest common divisor of two numbers may, in each system, be brought to an end by arriving at the remainder, zero; namely, in the system of natural numbers each remainder may be made less than the preceding one; in the biquadratic and cubic systems the *norm** of each remainder may be made less than that of the preceding remainder, and in the algebraic system the *degree* of each remainder may be made less than that of the preceding remainder.

The importance of the ending of Euclid's process lies in the fact that it determines a common divisor d of any two integers a, b, which is of the form $d = ax + by$ where x and y are integers. In particular, if a and b have no common divisors except units, then one may solve the indeterminate equation $ax + by = 1$ in integers x, y, whence one arrives at a demonstration of the fundamental principle of division, viz.

If two integers, a, b, have no common divisors except units, then every integer that is divisible by each is divisible by their product. For, since $ax + by = 1$ may be solved in integers x, y, any number c may be written $c = c(ax + by)$; whence, if $c = ma = nb$, we obtain $c = nb.ax + ma.by = ab(nx + my)$.

The fundamental problem in the arithmetical theory of any system of numbers is necessarily the establishment of this law of division. The more common form of the law is: "If a product be divisible by an integer a, and one factor of the product have with a no common divisors except units, then the other factor of the product is divisible by a." This form exhibits the analogy between the fundamental principle of arithmetic and that of algebra, which is: "If a product be equal to zero, and one factor of the product be finite, then the other factor of the product is equal to zero." These two principles lend themselves quite similarly in their respective subjects to the theory of the resolution of the quantities to which they apply into products of powers of irreducible factors.

*Norm of $a + b\sqrt{-1} = a^2 + b^2$, norm of $a + bt = a^2 - ab + b^2$.

3. After the development of the biquadratic and cubic theories, it became quite natural to expect similar results in the case of n^{ic} numbers or quantities of the form $a + bt + ct^2 + \ldots$, where a, b, c are natural numbers and t is a primitive n^{th} root of unity; but this expectation was not realized. An apparently insuperable obstacle was found in the failure of the fundamental principle of division. In 1847 Lamé gave a demonstration of the celebrated last theorem of Fermat,* based upon the properties of n^{ic} numbers. Liouville, however, pointed out that Lamé's demonstration is defective in that it assumes without proof that an n^{ic} number can be resolved, and in only one way, into a product of powers of irreducible n^{ic} numbers. Cauchy's attention was attracted to the subject by the discussion, and, failing to demonstrate, he assumed that the norm† of the remainder in the division of one n^{ic} number by another could be rendered less than the norm of the divisor. But the final result of this assumption was a *reductio ad absurdum.*

Meanwhile Kummer, Jacobi and others were at work on the generalization of the theorems of biquadratic and cubic reciprocity. Kummer thus found that the n^{ic} numbers ($n = 23$ or a higher prime) failed to conform to the fundamental law of division, and in order to overcome this difficulty he developed the theory of Ideal Primes. A synopsis of Kummer's theory, with references to his various memoirs on the subject, may be found in Smith's Report on the Theory of Numbers, B. A. Rep. 1860, pp. 120–40. The essential feature of the theory lies in the conception and definition of an ideal prime. In brief, if q be a natural prime that is resolvable into a product of true n^{ic} primes, and any n^{ic} number $f(t)$ be divisible by one of these primes, Kummer has shown that a corresponding congruence among natural numbers derivable from $f(t)$ is satisfied for the modulus q. Conversely, if q be not actually resolvable into a product of true primes, Kummer still considers it as *ideally* resolvable into a product of true primes corresponding to the congruences referred to. An ideal prime has thus, as its name implies, no actual existence among n^{ic} numbers, but one may determine whether or not an n^{ic} number $f(t)$ is divisible by such a prime by aid of the corresponding congruence, which is always actual.

4. Dirichlet next established an arithmetical theory of still more general

* See Smith's Report on the Theory of Numbers, B. A. Rep. 1860, p. 150.

† Norm of $a + bt + ct^2 + \ldots = (a + bt + ct^2 + \ldots)(a + bt' + ct'^2 + \ldots) \ldots$, t, t', ... being the primitive n^{th} roots of unity.

5. A few words with regard to the latest extensions in the theory of numbers will close this summary. Dedekind and Weber (Crelle, Vol. 92) have developed an arithmetical theory of complex functions of one variable x, or quantities of the form $a + bt + ct^2 + \ldots.$, where $a, b, c, \ldots.$ are rational entire functions of x, and t is a root of the irreducible equation $a_0 t^n + a_1 t^{n-1} + \ldots. = 0$, whose coefficients are rational entire functions of x. The *ideal* complex function, which is unavoidable in this case also, is identified, after Dedekind's method, by a corresponding Ideal, or assemblage of actual complex functions.

In the same volume (92) is a memoir by Kronecker, "Grundzüge einer arithmetischen Theorie der algebraischen Grossen," which treats of the arithmetical theory of complex functions of any number of variables. The subject is treated by Kronecker as Dirichlet treats the theory of complex numbers, viz. as a theory of forms.

6. In the present memoir the foundation is laid for an arithmetical theory of numbers in general. The argument is based upon the simplest laws of algebra, and the significance of the symbols of number $(a, b, c, \ldots.)$, as well as the significance of the ordinary symbols of algebra that connect them, $(=, +, -, \times, \div,)$, is entirely arbitrary so long as the required fundamental laws are conserved. A necessary concomitant of the theory is the introduction of ideal integers; and this introduction is accomplished after the method of Dedekind, by the preliminary establishment of a theory of ideals. The theory of ideals here established, however, differs from Dedekind's theory in important respects, chiefly in the definition of *relatively prime* ideals, and of the *product* of two ideals, and in the exclusion of certain kinds of ideals from the theory.

Part II is devoted to the consideration of fundamental principles and definitions, to a problem and the consequences of its solution, and to turning the question of ideal solution of the problem into the question of the establishment of a given theory of ideals. The propositions in this paper, although given in the order of their dependence upon one another, are, for the most part, left without demonstration; but the demonstrations are such as may be readily supplied by the reader.

Part III is devoted to a rigorous establishment of the theory of ideals that is indicated in Part II.

PART II.

1. (Definition.) *A Universe* (Körper, Rationalitäts-Bereich) is an aggregate of numbers a, b, c, ..., such that any combination of these numbers by *addition, subtraction, multiplication* and *division* (*i. e.* any *rational* combination) is a number of the aggregate.

We here use the word "number" in the sense of a general expression to denote the objects of our attention, a, b, c, ... These objects may be themselves aggregates of other objects or numbers from which our attention is for the time being abstracted. We use the words "addition," "subtraction," "multiplication" and "division," and also the word "equal," to indicate relations and operations that are represented in the usual way and conform to the usual axioms and laws of algebra. For example: "If equals be divided by equals, the quotients are equal": "$a(b+c)=ab+ac$," "$ab.c=a.bc$," "$ab=ba$," "$a^0=b^0=c^0, \ldots .=1$," "$a+a=2a$," etc.

2. (Definition.) An *Order* is an aggregate of numbers that contains every combination of its numbers by *addition, subtraction* and *multiplication* (*i. e.* every *rational entire* combination), and that also contains all the natural numbers 0, ± 1, ± 2, ± 3, ...

A universe may be regarded as divided, as to any one of its orders, into two divisions, the one consisting of numbers within the order, which may be called *integers*, the other consisting of numbers without the order, which may be called *fractions*. It is with the former numbers, or integers, that we are concerned in the present memoir.

3. (Definition.) An integer a is *divisible* by an integer b when the quotient ($b^{-1}a$ or a/b) is an integer, *i. e.* when $a=bc$, where c is an integer.

The phrase "*is* (or *are*) *divisible by*" occurs so frequently in the theory of numbers that we shall denote it by $\odot$, used by Prof. Sylvester.

4. (Theorem.) If $a \odot b$ and $b \odot c$, then $a \odot c$ and $a/c \odot b/c$.

5. (Theorem.) If a and $b \odot c$, then $ax+by \odot c$, where x, y are any integers.

6. (Definition.) A *Unit* is an integer that is a divisor of every integer.

In particular, a unit is a divisor of unity; and conversely, any integer that is a divisor of a unit is a unit. The products and quotients of units by units are also units.

7. (Definition.) *Associate integers* are integers that mutually divide each other.

Whence, the ratio of two associate integers is a unit; and if the ratio of two integers be a unit they are associates. Also, two or more associates of the same integer are associates of each other. Also, all associate integers divide, one the same integers as another, and are divisible by the same integers. We may thus select one integer out of a set of associate integers as typical of the whole, so far as questions of divisibility are concerned. A group of such integers corresponding to all sets of associate integers may be called a group of *Primary integers*, it being understood that these primary integers are always chosen (as they may be in an indefinite number of ways) so that the products of primary integers may be primary integers.

8. (Definition.) Two integers are *Relatively Prime* when every integer that is divisible by each of the two is divisible by their product.

(*a*). A unit is relatively prime to every integer.

(*b*). If a, b be relatively prime, then any two of their respective divisors are relatively prime.

(*c*). If each factor of one product be relatively prime to every factor of another product, then the two products are relatively prime.

E. g. let a be relatively prime to each α, β;

then if $\quad m\alpha\beta \odot a \;\therefore\; m\alpha\beta \odot a\alpha, \text{ or } m\beta \odot a,$ [8

$\therefore\; m \odot a \;\therefore\; m\alpha\beta \odot a\alpha\beta.$ Q. E. D. [8

(*d*). Two relatively prime integers have no common divisors except units.

9. (Problem 1.) To find a common divisor, d, of two integers, a, b, which is such that a/d, b/d are relatively prime.

We shall assume for the present that one can always find a solution of this problem in the case of any two integers a, b.*

10. (The fundamental principle of Division.) Two integers, a, b, that have no common divisors except units, are relatively prime.

* The term "relatively prime" used in the enunciation of Problem 1 may have various other definitions besides the one that we adopt, with the result that the fundamental principle of division may be established just as in the present instance, together with all the connected propositions. But such definition is then, for the order in question, equivalent to that of ¶ 8. On the other hand, Problem 1 is sometimes solvable according to no other definition than the one here chosen. This is the case, for example, in the order consisting of rational entire functions of an indeterminate x, whose coefficients are whole numbers only, not both whole numbers and fractions.

Euclid's process of finding the greatest common divisor of two numbers is a process for solving Problem 1 corresponding to the definition "Two integers, a, b, are relatively prime when the indeterminate equation $ax + by = 1$ may be solved in integers x, y." For Euclid's process finds, when it may be brought to an end, a common divisor, d, of a, b such that $d = ax + by$, where x, y are integers, *i. e.* such that $(a/d)\,x + (b/d)\,y = 1$.

18. (Theorem.) If no integer be divisible by an indefinitely great power of an integer that is not a unit, and if $a^n c \odot b^n$, however large n may be, then $a \odot b$.

For, let	d be a g. c. d. of a, b;	
then	$\because$ a^n/d^n, b^n/d^n are relatively prime,	[9, 8c
and	$\because$ $(a^n/d^n)/c \odot b^n/d^n$,	[hyp.
	$\therefore$ $c \odot b^n/d^n$,	[where n is indefinitely great
	$\therefore$ b/d is a unit,	[hyp.
	$\therefore$ $a \odot b$.	Q. E. D.

An order of such character that its integers may have infinite powers of non-unit integers as divisors may be called a *transcendental* order. For such orders this theorem is evidently not true. The property that "if $a^n c \odot b^n$, however large n may be, then $a \odot b$" may be taken, therefore, as a property pertaining to all non-transcendental orders for which the fundamental principle of division may be established. From this property one may derive that of ¶ 17; for it is this property upon which the solutions of Problems 1 and 2 will be made to depend (III. 17). Conversely, it will be shown hereafter that if no integer be divisible by an infinite number of non-unit integers, then from the property of ¶ 17 will follow the property just stated (III. 31).

19. (Definition.) A *Homogeneous* integer is an integer no two of whose divisors are relatively prime, except one be a unit.

(*a*). Any divisor of a homogeneous integer is a homogeneous integer.

(*b*). Of two divisors of a homogeneous integer, one is a divisor of the other.

(*c*). If neither of two integers be relatively prime to a given homogeneous integer, neither are they relatively prime to each other.

(*d*). If two homogeneous integers are relatively non-prime, any two of their respective divisors are relatively non-prime, except one be a unit.

(*e*). The product of two relatively non-prime homogeneous integers is a homogeneous integer.

(*f*). Of two relatively non-prime homogeneous integers, one is a divisor of the other.

20. (Theorem.) A primary integer cannot be resolved in more than one way into a product of primary relatively prime homogeneous integers.

When an order contains *primes*, *i. e.* integers that have no divisors except their associates and the units, then these primes and any powers of these primes

are homogeneous integers. We shall construct a system of integers hereafter, in which no primes exist, but in which homogeneous integers do exist.

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21. (On the question of an *Ideal* solution of Problem 1 or 2 in cases where no actual solution exists.)

Corresponding to each integer a there is an *assemblage* of integers, a, a', a'', ..., consisting of all integers that are divisible by a. This assemblage will be called the *Primary Ideal* corresponding to a, and will be denoted by $\mathfrak{a}$.

We may note that:

(a). A primary ideal that contains a', a'' also contains $a'x + a''y$, where x, y are any integers. [5

(b). The same primary ideal corresponds to any one of a set of associate integers, and those only.

(c). We may determine the significance of products, quotients, etc., of primary ideals by the condition that if $ab = c$, then $\mathfrak{a}\mathfrak{b} = \mathfrak{c}$, and if $a/b = c$, then $\mathfrak{a}/\mathfrak{b} = \mathfrak{c}$, etc.; so that the theory of division for primary ideals corresponds to that for integers.

Now, there may be assemblages of integers that satisfy the condition (a), but that do not correspond (as above) to any integers; *e. g.* $2x + (1 + \sqrt{-5})\,y$ in the order $a + b\sqrt{-5}$ (I. 4). If we can join any or all of these "ideals" with the primary ideals, and extend to the whole an arithmetical theory based upon that already determined for the primary ideals, and in which Problem 1 or 2 shall always have a solution, then it is plain that this theory may be turned again into a theory of integers by regarding the non-primary ideals as corresponding to sets of associate *ideal* integers*; and that in this theory Problems 1 and 2 have always actual or ideal solutions.

It happens that there are various ways of generalizing the theory of primary ideals. Dedekind, for instance, joins with the primary ideals all other ideals. He further defines the product of two or more ideals as the assemblage of all products and sums of products of the integers of the respective factors; this being a definition that coincides with (c) when the ideals are primary (Hauptideale).

* By definition a unit must pertain as divisor to the assemblage of all integers of the order. As this assemblage is a primary ideal, no ideal integer can be a unit. A set of associate ideal integers consists therefore of the products of an assumed *primary* ideal integer (7) into all units of the order.

There are examples, however, that show the impossibility of reaching the utmost limits of generalization by the direct definition of the product and the consideration of all ideals. We may learn what are the most general features of the extended theory of ideals by assuming the theory of actual and ideal integers to be properly established, and deriving from that theory the theory of the corresponding, or *proper*, ideals, so as to obtain the following results:

(*d*). If m be the least common multiple of a, b, then $\mathfrak{m}$ shall consist of all integers that are common to $\mathfrak{a}$, $\mathfrak{b}$.

(*e*). If a be divisible by b, then $\mathfrak{a}$ is contained in $\mathfrak{b}$; and conversely.

(*f*). If a' be an actual integer, then shall $\mathfrak{a}'\mathfrak{b}$ be the assemblage of all products of a' into the integers of $\mathfrak{b}$, viz. $\mathfrak{a}'\mathfrak{b} = a'\mathfrak{b}$.

(*g*). The quotient $\mathfrak{a}/\mathfrak{b}$ shall consist of all integers c' such that $c'\mathfrak{b}$ is contained in $\mathfrak{a}$.

(*h*). The product $\mathfrak{a}\mathfrak{b}$ is the quotient of $a'\mathfrak{b}$ by $\mathfrak{c}$, where a' is an integer of $\mathfrak{a}$ and $\mathfrak{a}'/\mathfrak{a} = \mathfrak{c}$.

With these suggestions as to the theory of ideals, I have established its existence, using the property of ¶ 18, and the restriction that all proper ideals shall be obtainable from the primary ideals by successive derivations of least common multiples and quotients. The theory which we proceed to give rests, however, upon a different and possibly more general definition of integer ideals.

PART III.

In what follows, the integers referred to are supposed to be those of some order that is not specified except in the case of examples.

1. (Definition.) An *Ideal* of an order is such an assemblage of integers that if a', a'' be any two integers of the assemblage, then $a'x + a''y$ is an integer of the assemblage; x, y being any integers of the order.

Ideals will be represented by old English letters; thus: $\mathfrak{a}$, $\mathfrak{b}$, ... The corresponding italic letters, supplied with accents or subscripts, will denote integers of these ideals; thus: a', a'', a_1, a_2, ... are integers of $\mathfrak{a}$.

The ideal which consists of all products of an integer k into the integers of an ideal $\mathfrak{a}$ will be denoted by $k\mathfrak{a}$ or $\mathfrak{a}k$.

2. (Definition.) An ideal $\mathfrak{a}$ is *divisible* by an ideal $\mathfrak{b}$ when every integer of $\mathfrak{a}$ is an integer of $\mathfrak{b}$.

The symbol $\odot$ will be used to denote that one ideal is divisible by another, just as with integers. This symbol, when used between ideals, is thus equivalent to the words "consists of integers of."

(*a*). If $\mathfrak{a} \odot \mathfrak{b}$ and $\mathfrak{b} \odot \mathfrak{a}$, then $\mathfrak{a} = \mathfrak{b}$.

(*b*). If $\mathfrak{a} \odot \mathfrak{b}$ and $\mathfrak{b} \odot \mathfrak{c}$, then $\mathfrak{a} \odot \mathfrak{c}$.

(*c*). $a'\mathfrak{b} \odot \mathfrak{a}$; $b'\mathfrak{a} \odot \mathfrak{a}$.

(*d*). If $\mathfrak{a} \odot \mathfrak{b}$, then $k\mathfrak{a} \odot k\mathfrak{b}$.

(*e*). If $k\mathfrak{a} \odot k\mathfrak{b}$, then $\mathfrak{a} \odot \mathfrak{b}$.

The aggregate of all integers, in the order that we are considering, forms *the unit* ideal, and will be denoted by $\mathfrak{o}$. A *primary* ideal is one that consists of all integers divisible by a given integer of the order. Evidently $k\mathfrak{o}$ is the primary ideal that *corresponds* to k.

We shall make use of any integer k in a sense that will be equivalent to the use of $k\mathfrak{o}$; *i. e.* k shall be divisible by an ideal when it is an integer of that ideal, and k shall be a divisor of an ideal when it is a divisor of every integer of that ideal.

(*f*). If $\mathfrak{a} \odot k$, then $\mathfrak{a} = k\mathfrak{b}$, where $\mathfrak{b}$ is an ideal.

3. (Definition.) The *Least Common Multiple* of two or more ideals is the assemblage of all their common integers.

(*a*). The least common multiple of two or more ideals is an ideal. For if

m', m'' be common integers of $\mathfrak{a}, \mathfrak{b}, \ldots\ldots,$

then $m'x + m''y$ is a common integer of $\mathfrak{a}, \mathfrak{b}, \ldots\ldots$ Q. E. D. [1

(*b*). The least common multiple of two or more ideals is divisible by each ideal.

(*c*). An ideal that is divisible by each of two or more ideals is divisible by their least common multiple.

4. (Definition.) The *Quotient* $\mathfrak{a}/\mathfrak{b}$ of an ideal $\mathfrak{a}$ by an ideal $\mathfrak{b}$ consists of all integers k such that $k\mathfrak{b} \odot \mathfrak{a}$.

(Rem.) If $\mathfrak{a}$ be not divisible by $\mathfrak{b}$, then $\mathfrak{a}/\mathfrak{b}$ is the same as $\mathfrak{m}/\mathfrak{b}$ where $\mathfrak{m}$ is the least common multiple of $\mathfrak{a}, \mathfrak{b}$. For, $\because k\mathfrak{b} \odot \mathfrak{a}, \mathfrak{b}, \therefore k\mathfrak{b} \odot \mathfrak{m}$ or $k \odot \mathfrak{m}/\mathfrak{b}$; and conversely, if k' be any integer of $\mathfrak{m}/\mathfrak{b}$, then $\because k'\mathfrak{b} \odot \mathfrak{m} \odot \mathfrak{a}$ $\therefore k' \odot \mathfrak{a}/\mathfrak{b}$ $\therefore \mathfrak{m}/\mathfrak{b} = \mathfrak{a}/\mathfrak{b}$.

The assemblage of such integers k that $k\mathfrak{b} \odot \mathfrak{a}$ need not therefore be represented by $\mathfrak{a}/\mathfrak{b}$ when $\mathfrak{a}$ is not divisible by $\mathfrak{b}$, but by $\mathfrak{m}/\mathfrak{b}$. So far as the

present investigation is concerned, the symbol $\mathfrak{a}/\mathfrak{b}$ shall have no meaning, and will not be employed, unless $\mathfrak{a} \odot \mathfrak{b}$.

(*a*). The quotient of $\mathfrak{a}$ by $\mathfrak{b}$ is an ideal.

For if c', c'' be any two integers of the quotient,

then $\because\ c'b',\ c''b'$ are integers of $\mathfrak{a}$, [4

$\therefore\ c'b'x + c''b'y$ or $(c'x + c''y)\,b' \odot \mathfrak{a}$, or $(c'x + c''y)\,\mathfrak{b} \odot \mathfrak{a}$. Q. E. D.

(*b*). $\mathfrak{a} \odot \mathfrak{a}/\mathfrak{b}$.

For $\because\ a'\mathfrak{b} \odot \mathfrak{a}\ \therefore\ \mathfrak{a} \odot \mathfrak{a}/\mathfrak{b}$. Q. E. D. [2*c*, 4

(*c*). If $\mathfrak{a}/\mathfrak{b} = \mathfrak{c}$, then $b'\mathfrak{c} \odot \mathfrak{a}$; *i. e.*, $b'(\mathfrak{a}/\mathfrak{b}) \odot \mathfrak{a}$.

(*d*). If $\mathfrak{a} \odot \mathfrak{b} \odot \mathfrak{c}$, then $\mathfrak{a}/\mathfrak{c} \odot \mathfrak{b}/\mathfrak{c}$ and $\mathfrak{a}/\mathfrak{c} \odot \mathfrak{a}/\mathfrak{b}$.

For if k be any integer of $\mathfrak{a}/\mathfrak{c}$,

then $\because\ k\mathfrak{c} \odot \mathfrak{a} \odot \mathfrak{b}\ \therefore\ k \odot \mathfrak{b}/\mathfrak{c}$, Q. E. D. [4

and $\because\ k\mathfrak{b} \odot k\mathfrak{c} \odot \mathfrak{a}\ \therefore\ k \odot \mathfrak{a}/\mathfrak{b}$. Q. E. D. [2*d*, 4

(*e*). $k\mathfrak{a}/k\mathfrak{b} = \mathfrak{a}/\mathfrak{b}$.

For let $\mathfrak{c} = k\mathfrak{a}/k\mathfrak{b}$, $\mathfrak{d} = \mathfrak{a}/\mathfrak{b}$; [2*e*

then $\because\ d'\mathfrak{b} \odot \mathfrak{a}\ \therefore\ d'k\mathfrak{b} \odot k\mathfrak{a}\ \therefore\ \mathfrak{d} \odot k\mathfrak{a}/k\mathfrak{b} = \mathfrak{c}$, [4, 2*d*

and $\because\ c'k\mathfrak{b} \odot k\mathfrak{a}\ \therefore\ c'\mathfrak{b} \odot \mathfrak{a}\ \therefore\ \mathfrak{c} \odot \mathfrak{a}/\mathfrak{b} = \mathfrak{d}$, [2*e*

$\therefore\ \mathfrak{c} = \mathfrak{d}$. Q. E. D.

(*f*). $k\mathfrak{a}/k = \mathfrak{a}/\mathfrak{o} = \mathfrak{a}$.

(*g*). $k(\mathfrak{a}/k) = \mathfrak{a}$. (Put $\mathfrak{a} = k\mathfrak{b}$.) [2*f*, 4*f*

5. (Definition.) The *Product* $\mathfrak{a}\mathfrak{b}$ of two ideals $\mathfrak{a}$, $\mathfrak{b}$ is any ideal $\mathfrak{w}$, if it may be found, such that $\mathfrak{w}/\mathfrak{a} = \mathfrak{b}$.

(Rem.) In general there may be more than one product of two ideals, or there may be none; also, $\mathfrak{a}\mathfrak{b}$ and $\mathfrak{b}\mathfrak{a}$ may differ.

The problem before us is to show that, by considering an order whose integers satisfy a given condition, we may determine an aggregate of ideals of that order whose products are associative, commutative, unambiguous, and within the aggregate. There may perhaps be more than one such aggregate; but all are included within the aggregate of ideals, which are defined as follows:

6. (Definition.)* A *Proper* Ideal $\mathfrak{a}$ is one such that if $\mathfrak{t}k \odot \mathfrak{a}$ and $\mathfrak{t}k/\mathfrak{a} \odot \mathfrak{t}$, then $k \odot \mathfrak{a}$.

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* Either of the following definitions of proper ideals is equivalent to this definition: "A *Proper* ideal $\mathfrak{a}$ is one such that if $\mathfrak{w}/\mathfrak{a} = \mathfrak{b}$, then $\mathfrak{w}/\mathfrak{b} = \mathfrak{a}$." "A *Proper* ideal $\mathfrak{a}$ is one such that it is the *quotient* of a primary ideal $a'\mathfrak{o}$ by some ideal $\mathfrak{b}$."

(a). If $\mathfrak{a}$ be a proper ideal, and if $k\mathfrak{t} \odot \mathfrak{a}$ and $k\mathfrak{t}/\mathfrak{a} \odot k$, then $\mathfrak{t} \odot \mathfrak{a}$.

7. (Theorem.) The least common multiple, $\mathfrak{m}$, of two or more proper ideals, $\mathfrak{a}, \mathfrak{b}, \ldots.$, is a proper ideal.

For let k be any integer such that $\mathfrak{t}k \odot \mathfrak{m}$ and $\mathfrak{t}k/\mathfrak{m} \odot \mathfrak{t}$;

then $\because\ \mathfrak{t}k \odot \mathfrak{m} \odot \mathfrak{a}\ \therefore\ \mathfrak{t}k/\mathfrak{a} \odot \mathfrak{t}k/\mathfrak{m} \odot \mathfrak{t}$, [4$d$

$\therefore\ k \odot \mathfrak{a}$. So, $k \odot \mathfrak{b}, \ldots.$ [$\mathfrak{a}, \mathfrak{b}, \ldots.$ proper ideals

$\therefore\ k \odot \mathfrak{m}$. Q. E. D.

8. (Theorem.) The quotient, $\mathfrak{c}$, of a proper ideal $\mathfrak{a}$ by an ideal $\mathfrak{b}$ is a proper ideal.

For let k be any integer such that $\mathfrak{t}k \odot \mathfrak{c}$ and $\mathfrak{t}k/\mathfrak{c} \odot \mathfrak{t}$;

then $\because\ \mathfrak{t}k \odot \mathfrak{c}\ \therefore\ \mathfrak{t}b'k \odot b'\mathfrak{c} \odot \mathfrak{a}$, [2$d$, 4$c$

$\therefore\ \mathfrak{t}b'k/\mathfrak{a} \odot \mathfrak{t}k/\mathfrak{c} \odot \mathfrak{t}$, [4$d$, 4$e$

$\therefore\ b'k \odot \mathfrak{a}$, [$\mathfrak{a}$ a proper ideal

$\therefore\ k\mathfrak{b} \odot \mathfrak{a}\ \therefore\ k \odot \mathfrak{c}$. Q. E. D.

9. (Theorem.) If $\mathfrak{a}/\mathfrak{b} = \mathfrak{c}$, where $\mathfrak{b}$ is a proper ideal, then $\mathfrak{a}/\mathfrak{c} = \mathfrak{b}$.

For let k be any integer of $\mathfrak{a}/\mathfrak{c}$;

then $\because\ \mathfrak{c}k \odot \mathfrak{a} \odot \mathfrak{b}\ \therefore\ \mathfrak{c}k/\mathfrak{b} \odot \mathfrak{a}/\mathfrak{b} = \mathfrak{c}$, [4$d$

$\therefore\ k \odot \mathfrak{b}$ or $\mathfrak{a}/\mathfrak{c} \odot \mathfrak{b}$, [$\mathfrak{b}$ a proper ideal

and $\because\ b'\mathfrak{c} \odot \mathfrak{a}\ \therefore\ \mathfrak{b} \odot \mathfrak{a}/\mathfrak{c}$, [4$c$

$\therefore\ \mathfrak{a}/\mathfrak{c} = \mathfrak{b}$. Q. E. D.

10. (Cor.) If $\mathfrak{b}$ and $\mathfrak{c}$ be proper ideals, then $\mathfrak{b}\mathfrak{c} = \mathfrak{c}\mathfrak{b}$; *i. e.* every dividend for which $\mathfrak{b}$ is divisor and $\mathfrak{c}$ quotient is a dividend for which $\mathfrak{c}$ is divisor and $\mathfrak{b}$ quotient, and *vice versa*.

11. (Theorem.) If $(\mathfrak{w}/\mathfrak{a})/\mathfrak{b} = \mathfrak{c}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are proper ideals, then $(\mathfrak{w}/\mathfrak{c})/\mathfrak{a} = \mathfrak{b}$.

For $\because\ \mathfrak{w} \odot \mathfrak{w}/\mathfrak{a} \odot \mathfrak{c}\ \therefore\ \mathfrak{w}/\mathfrak{c} \odot \mathfrak{w}/(\mathfrak{w}/\mathfrak{a}) = \mathfrak{a}$. [4$b$, 4$d$, 9

Hence, let k be any integer of $(\mathfrak{w}/\mathfrak{c})/\mathfrak{a}$;

then $\because\ k\mathfrak{a} \odot \mathfrak{w}/\mathfrak{c}\ \therefore\ kc'\mathfrak{a} \odot \mathfrak{w}\ \therefore\ kc' \odot \mathfrak{w}/\mathfrak{a}$, [4, 2$d$, 4$c$

$\therefore\ k\mathfrak{c} \odot \mathfrak{w}/\mathfrak{a}\ \therefore\ k \odot (\mathfrak{w}/\mathfrak{a})/\mathfrak{c} = \mathfrak{b}$, [9

and $\because\ b'\mathfrak{c} \odot \mathfrak{w}/\mathfrak{a}\ \therefore\ b'a'\mathfrak{c} \odot \mathfrak{w}\ \therefore\ b'a' \odot \mathfrak{w}/\mathfrak{c}$, [4$c$, 2$d$

$\therefore\ b'\mathfrak{a} \odot \mathfrak{w}/\mathfrak{c}\ \therefore\ \mathfrak{b} \odot (\mathfrak{w}/\mathfrak{c})/\mathfrak{a}$,

$\therefore\ (\mathfrak{w}/\mathfrak{c})/\mathfrak{a} = \mathfrak{b}$. Q. E. D.

12. (Cor.) The existence of a product of two proper ideals, $\mathfrak{a}$ and $\mathfrak{b}$, depends upon the existence of an ideal $\mathfrak{w}$ such that $\mathfrak{w} \odot \mathfrak{a}$ and $\mathfrak{w}/\mathfrak{a} \odot \mathfrak{b}$ [viz.

from $(\mathfrak{w}/\mathfrak{a})/\mathfrak{b}=\mathfrak{c}$ one finds $\mathfrak{w}/\mathfrak{c}=\mathfrak{ab}$ (11)]; and if $\mathfrak{w}$ be a proper ideal, then the corresponding product, $\mathfrak{ab}$, is a proper ideal. [8

13. (Cor.) If $\mathfrak{a}$, $\mathfrak{b}$ and $(\mathfrak{w}/\mathfrak{a})/\mathfrak{b}=\mathfrak{c}$ be proper ideals, then

$$\mathfrak{w}/\mathfrak{a}=\mathfrak{bc}=\mathfrak{cb},\ \mathfrak{w}/\mathfrak{b}=\mathfrak{ac}=\mathfrak{ca},\ \mathfrak{w}/\mathfrak{c}=\mathfrak{ab}=\mathfrak{ba},\ (\mathfrak{w}/\mathfrak{a})/\mathfrak{b}=\mathfrak{w}/\mathfrak{ab}=\mathfrak{c},\ldots.;$$

i. e. $\mathfrak{a}.\mathfrak{bc}=\mathfrak{a}.\mathfrak{cb}=\mathfrak{b}.\mathfrak{ca}=\mathfrak{b}.\mathfrak{ac}=\mathfrak{c}.\mathfrak{ab}=\mathfrak{c}.\mathfrak{ba}=\mathfrak{ab}.\mathfrak{c},\ldots.$

14. (Cor.) Multiplication among proper ideals is associative and commutative; *i. e.* every ideal that is a product of given proper ideals corresponding to one arrangement and association of the factors is also a product corresponding to any other arrangement and association of the factors.

15. (Theorem.) If $\mathfrak{a}/\mathfrak{b}\odot\mathfrak{a}/\mathfrak{c}$, where $\mathfrak{b}$ is a proper ideal, then $\mathfrak{c}\odot\mathfrak{b}$.

For let $\mathfrak{a}/\mathfrak{c}=\mathfrak{v}$;

then $\because c'\mathfrak{v}\odot\mathfrak{a}\ \therefore\ \mathfrak{c}\odot\mathfrak{a}/\mathfrak{v}$,

and $\because \mathfrak{a}\odot\mathfrak{a}/\mathfrak{b}\odot\mathfrak{v}\ \therefore\ \mathfrak{a}/\mathfrak{v}\odot\mathfrak{a}/(\mathfrak{a}/\mathfrak{b})=\mathfrak{b}$, [4*d*, 9

$\therefore\ \mathfrak{c}\odot\mathfrak{a}/\mathfrak{v}\odot\mathfrak{b}$. Q. E. D.

16. (Cor.) If $\mathfrak{a}/\mathfrak{b}=\mathfrak{a}/\mathfrak{c}$, where $\mathfrak{b}$ and $\mathfrak{c}$ are proper ideals, then $\mathfrak{b}=\mathfrak{c}$.

17. (Definition.) A *System* is an order whose integers a, b, c, ... are such that if $a^n c\odot b^n$, however large n may be, then $a\odot b$. [Compare Part II, 18

The propositions that follow have reference only to integers of a system.

18. (Lemma.) If $k\mathfrak{t}\odot a\mathfrak{t}$, then $k\odot a$.

For $\because k\mathfrak{t}\odot a\mathfrak{t}\ \therefore\ kt'=at_1$;

so $kt_1=at_2,\ kt_2=at_3,\ \ldots.,\ kt_{n-1}=at_n,$ [$n=1, 2, 3, \ldots.$ *ad inf.*

$\therefore\ k^n t' t_1 t_2 \ldots. t_{n-1}=a^n t_1 t_2 \ldots. t_n\ \therefore\ k^n t'=a^n t_n$,

$\therefore\ k^n t'\odot a^n$, however large n may be; $\therefore\ k\odot a$. Q. E. D. [17

19. (Cor.) If $k\mathfrak{v}\odot a\mathfrak{t}$ where $\mathfrak{t}\odot\mathfrak{v}$, then $k\odot a$.

For $\because \mathfrak{t}\odot\mathfrak{v}\ \therefore\ k\mathfrak{t}\odot k\mathfrak{v}\odot a\mathfrak{t}\ \therefore\ k\odot a$.

20. (Cor.) A primary ideal is a proper ideal.

For let $a\mathfrak{o}$ be a primary ideal, and let k be an integer such that $\mathfrak{t}k\odot a$ and $\mathfrak{t}k/a\odot\mathfrak{t}$;

then $\because \mathfrak{t}k/a\odot\mathfrak{t}\ \therefore\ \mathfrak{t}k\odot\mathfrak{t}a$, [4*g*

$\therefore\ k\odot a$. Q. E. D.

21. (Theorem.) If $\mathfrak{a}\odot\mathfrak{b}$, then $k\mathfrak{a}/\mathfrak{b}=k(\mathfrak{a}/\mathfrak{b})$.

For let $\mathfrak{u}=k\mathfrak{a}/\mathfrak{b}$, $\mathfrak{v}=\mathfrak{a}/\mathfrak{b}$;

then $\because u\mathfrak{b}' \odot k\mathfrak{a}$ and $\because \mathfrak{a} \odot \mathfrak{b} \ \therefore u' \odot k,$ [19

$\therefore (u'/k)\mathfrak{b} \odot \mathfrak{a} \ \therefore u'/k \odot \mathfrak{a}/\mathfrak{b} \ \therefore \mathfrak{u} \odot k(\mathfrak{a}/\mathfrak{b}) = k\mathfrak{v},$ [2*e*, 2*d*

and $\because v'\mathfrak{b} \odot \mathfrak{a} \ \therefore kv'\mathfrak{b} \odot k\mathfrak{a} \ \therefore k\mathfrak{v} \odot k\mathfrak{a}/\mathfrak{b} = \mathfrak{u},$

$\therefore \mathfrak{u} = k\mathfrak{v}.$ Q. E. D.

22. (Cor.) There exists, corresponding to any two proper ideals $\mathfrak{a}$ and $\mathfrak{b}$, a proper ideal that is their product.

For $\because a'b'\mathfrak{o}$ is a proper ideal divisible by $\mathfrak{a}$,

and $\because a'b'/\mathfrak{a} = b'(a'/\mathfrak{a}) \odot \mathfrak{b},$ [21, 2*c*

$\therefore a'b'$ by $(a'b'/\mathfrak{a})/\mathfrak{b}$ is a proper ideal $\mathfrak{ab}$. Q. E. D. [12

23. (Cor.) If $\mathfrak{a}$ be a proper ideal, then $k\mathfrak{a}$ is the proper ideal $k\mathfrak{o}/\mathfrak{a}$.

24. (Theorem.) If $\mathfrak{a}/\mathfrak{c} \odot \mathfrak{b}/\mathfrak{c}$, where $\mathfrak{b}$ is a proper ideal, then $\mathfrak{a} \odot \mathfrak{b}$.

For let k be any integer of $b'\mathfrak{a}/\mathfrak{b}$;

then $\because k\mathfrak{b} \odot b'\mathfrak{a} \odot \mathfrak{c}$ and $\because \mathfrak{a} \odot \mathfrak{c},\ \mathfrak{b} \odot \mathfrak{c} \ \therefore k(\mathfrak{b}/\mathfrak{c}) \odot b'(\mathfrak{a}/\mathfrak{c}),$ [21, 4*d*

and $\because \mathfrak{a}/\mathfrak{c} \odot \mathfrak{b}/\mathfrak{c} \ \therefore k \odot b'$, *i. e.* $b'\mathfrak{a}/\mathfrak{b} \odot b',$ [19

$\therefore \mathfrak{a} \odot \mathfrak{b}.$ Q. E. D. [6*a*

25. (Cor.) If $\mathfrak{a}/\mathfrak{c} = \mathfrak{b}/\mathfrak{c}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are proper ideals, then $\mathfrak{a} = \mathfrak{b}$.

26. (Recapitulation.) The proper ideals of a system are duplicated by multiplication amongst themselves (22); multiplication and division conform to the ordinary rules of algebra (14); of the three related elements, *product, divisor, quotient*, any two being given, the other is unambiguously determined (4, 25, 16), while $\mathfrak{a} \odot \mathfrak{b}$ implies that $\mathfrak{a} = \mathfrak{bc}$, and conversely, wherein $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are proper ideals.

27. (Definition.) Two ideals are Relatively Prime when every ideal* that is divisible by each is divisible by their product.

28. (Solution of Problem 2.) If $\mathfrak{m}$ be the least common multiple of the proper ideals $\mathfrak{a}$ and $\mathfrak{b}$, then $\mathfrak{m}/\mathfrak{a}$ and $\mathfrak{m}/\mathfrak{b}$ are relatively prime.

For let k be any integer divisible by $\mathfrak{m}/\mathfrak{a}$ and $\mathfrak{m}/\mathfrak{b}$,

then $\because k\mathfrak{a}, k\mathfrak{b} \odot \mathfrak{m} \ \therefore k\mathfrak{ab} \odot \mathfrak{ma}, \mathfrak{mb} \ \therefore k\mathfrak{ab}/\mathfrak{m} \odot \mathfrak{a}, \mathfrak{b},$

$\therefore k\mathfrak{ab}/\mathfrak{m} \odot \mathfrak{m} \ \therefore k \odot \mathfrak{m}^2/\mathfrak{ab}.$ Q. E. D.

(Conclusion.) If the primary divisor, *actual* or *ideal*, of all integers of an integer ideal $\mathfrak{a}$, and those only, be denoted by a, then such system of actual and ideal integers fulfils the fundamental principle of division. Q. E. D. [II, 16, 9, 10

* "Proper" or "primary ideal" may be substituted for the word "ideal" in this definition without loss of generality.

29. (Theorem.) If $\mathfrak{b}$ be a proper ideal and $\mathfrak{a}^n\mathfrak{c} \odot \mathfrak{b}^n$, however large n may be, then $\mathfrak{a} \odot \mathfrak{b}$.

For let $b'/\mathfrak{b} = \mathfrak{d}$;

then $\quad \because\ a'^n c' \odot \mathfrak{a}^n\mathfrak{c} \odot \mathfrak{b}^n,$

$\therefore\ (a'd')^n c' \odot (d'\mathfrak{b})^n \odot b'^n \ \therefore\ a'd' \odot b',$ [17

$\therefore\ \mathfrak{a} \odot b'/\mathfrak{d} = \mathfrak{b}.$ Q. E. D.

30. (Cor.) No ideal is divisible by an indefinitely great power of any other proper ideal than $\mathfrak{o}$.

For if $\mathfrak{c},\ = \mathfrak{o}^n\mathfrak{c},\ \odot \mathfrak{b}^n$, however large n may be,

$\therefore\ \mathfrak{o} \odot \mathfrak{b}\ \therefore\ \mathfrak{b} = \mathfrak{o}.$ Q. E. D.

31. (Theorem). Any order is a system in which no ideal is divisible by an infinite number of ideals, and in which no fractional number a/b satisfies an equation $x^n + a_1x^{n-1} + \ldots\ a = 0$, whose coefficients, $a_1, \ldots a_n$, are integers of the order.

For let $a^n c \odot b^n$; *i. e.*, let $cx = c(a/b)$, $cx^2, \ldots cx^n$ be integers, however large n may be. From the integers $cx, \ldots$ form the series of ideals (cx, cx^2), $(cx, cx^2, cx^3), \ldots (cx, cx^2, \ldots cx^n)$, where, to explain the notation, the first ideal consists of all integers of the form $cx.u + cx^2.v$, u, v being any integers of the order; then

$\because$ the first of the infinite number of ideals so formed is divisible by all the others,

$\therefore$ the series consists of a finite number of different ideals, [hyp.

$\therefore$ by taking s large enough we shall have

$$(cx, cx^2, \ldots cx^s) = (cx, cx^2, \ldots cx^t), \text{ where } t < s;$$

$\therefore\ cx^s = cxu_1 + cx^2u_2 + \ldots cx^tu_t$, where $u_1, \ldots u_t$ are integers of the order,

$\therefore\ x^s - u_tx^t - u_{t-1}x^{t-1} - \ldots - u_1x = 0,$

$\therefore\ x$ is not a fractional number; *i. e.*, $a \odot b$. Q. E. D. [hyp.

32. (Definition.) The *Compound* of two or more ideals, $\mathfrak{a}, \mathfrak{b}, \ldots$ is the ideal consisting of all products, $a'b', \ldots$ between the integers of the given ideals and the sums of those products.

The compound of $\mathfrak{a}, \mathfrak{b} \ldots$ will be denoted by $\mathfrak{a} \times \mathfrak{b} \times \ldots$; the compound of $\mathfrak{a}$, taken n times, by $[\mathfrak{a}]^n$.

(*a*). $\mathfrak{a}\times\mathfrak{b}\times\mathfrak{c} = \mathfrak{b}\times\mathfrak{a}\times\mathfrak{c} = \ldots = \mathfrak{a}\times(\mathfrak{b}\times\mathfrak{c}) = \mathfrak{b}\times(\mathfrak{a}\times\mathfrak{c}) = \ldots;$

$$[\mathfrak{a}\times\mathfrak{b}]^n = [\mathfrak{a}]^n \times [\mathfrak{b}]^n.$$

(*b*). The compound of two or more ideals is divisible by each of the ideals.

(*c*). The compound of divisor and quotient is divisible by the dividend; also, the compound of any number of ideals is divisible by any product of those ideals.

(*d*). If $\mathfrak{a} \odot \mathfrak{b}$ and $\mathfrak{x} \odot \mathfrak{y}$, then $\mathfrak{a} \times \mathfrak{x} \odot \mathfrak{b} \times \mathfrak{y}$; in particular $[\mathfrak{a}]^n \odot [\mathfrak{b}]^n$.

(*e*). If there be any ideal $\mathfrak{c}$ such that $\mathfrak{c}/\mathfrak{a} = \mathfrak{b}$, then $\mathfrak{a} \times \mathfrak{b}$ is such an ideal.

For $\because \mathfrak{a} \times \mathfrak{b} \odot \mathfrak{c} \ \therefore \mathfrak{a} \times \mathfrak{b}/\mathfrak{a} \odot \mathfrak{c}/\mathfrak{a} = \mathfrak{b}$, [32*c*, 4*d*, hyp.

and $\because \mathfrak{b} \times \mathfrak{a} \odot \mathfrak{a} \times \mathfrak{b} \ \therefore \mathfrak{b} \odot \mathfrak{a} \times \mathfrak{b}/\mathfrak{a}$, [4

$\therefore \mathfrak{a} \times \mathfrak{b}/\mathfrak{a} = \mathfrak{b}$. Q. E. D.

(*f*). The product of two or more proper ideals may be obtained by compounding them, when such compound is a proper ideal.

(*g*). If $\mathfrak{b}$ be a proper ideal and $[\mathfrak{a}]^n \times \mathfrak{c} \odot [\mathfrak{b}]^n$, however large n may be, then $\mathfrak{a} \odot \mathfrak{b}$.

For let $b'/\mathfrak{b} = \mathfrak{d}$;

then $\because a'^n c' \odot [\mathfrak{a}]^n \times \mathfrak{c} \odot [\mathfrak{b}]^n$, [hyp.

$\therefore (a'd')^n c' \odot [d'\mathfrak{b}]^n \odot b'^n \ \therefore a'd' \odot b'$, [32*a*, 32*c*, 17

$\therefore \mathfrak{a} \odot b'/\mathfrak{d} = \mathfrak{b}$. Q. E. D.

CORNELL UNIVERSITY, June, 1886.

A Theorem respecting the Singularities of Curves of Multiple Curvature.

By Henry B. Fine.

In a paper which appeared in Vol. VIII, No. 2, of this Journal, I showed that any singularity, however complex, possible to an element of a point curve of double curvature may be completely defined by three singularity indices, $\kappa_1, \kappa_2, \kappa_3$. If A_0 be the element and $P_t = A_0 + A_1 t + A_2 t^2 +$ etc. be the Grassmann equation of the curve for its neighborhood, κ_1 is the number of consecutive coefficients A_1, A_2, etc., congruent with A_0; κ_2, the number of consecutive coefficients $A_{\kappa_1+2}, \ldots.$ linearly derivable from A_0 and A_{κ_1+1}; κ_3, the number of consecutive coefficients $A_{\kappa_1+\kappa_2+3} \ldots.$ linearly derivable from $A_0, A_{\kappa_1+1}, A_{\kappa_1+\kappa_2+2}$. The point P_{dt} is then distant an infinitesimal of the order $\kappa_1 + 1$ from A_0, an infinitesimal of the order $\kappa_1 + \kappa_2 + 2$ from $A_0 A_{\kappa_1+1}$, the tangent at A_0, and an infinitesimal of the order $\kappa_1 + \kappa_2 + \kappa_3 + 3$ from $A_0 A_{\kappa_1+1}, A_{\kappa_1+\kappa_2+2}$, the osculating plane at A_0. In other words, the point A_0 of the singularity indices $\kappa_1, \kappa_2, \kappa_3$ is stationary to the degree κ_1, a point of contact of the order $\kappa_1 + \kappa_2 + 1$ of the curve with its tangent, and a point of contact of the order $\kappa_1 + \kappa_2 + \kappa_3 + 1$ of the curve with its osculating plane.

The singularities of curves of any degree of curvature whatsoever admit of similar treatment. Besides the singularities of which $\kappa_1, \kappa_2, \kappa_3$ measure the degree, and which may be conveniently called singularities of the 1st, 2d and 3d class respectively, there will be possible to a curve of $(n-1)^{\text{ple}}$ curvature singularities of a 4th, 5th, ... nth class also, in consequence of the n dimensionality of the space supposed. Let indices $\kappa_4 \ldots. \kappa_n$ measure the degrees of these additional singularities, and $\kappa_1 \ldots. \kappa_n$ will completely define the singularity of any curve point A_0. κ_4 represents the number of consecutive coefficients $A_{\kappa_1+\kappa_2+\kappa_3+4} \ldots.$ in the Grassmann equation, which are linearly derivable from $A_0, A_{\kappa_1+1}, A_{\kappa_1+\kappa_2+2}, A_{\kappa_1+\kappa_2+\kappa_3+3}$; and κ_5, etc., have analogous meanings.

A curve of $(n-1)^{\text{ple}}$ curvature will be touched at every point, not only by a definite line and plane, but, when $n > 3$, by a definite space of 3 dimensions, one of 4 dimensions, and so on to one of $n-1$ dimensions also; the tangent S_3 at any curve point A_0, for instance, being the S_3 determined by any four curve points P_1, P_2, P_3, P_4 in what is its limiting position when P_1, P_2, P_3, P_4 are made, independently of each other, to approach A_0. And the point A_0 of singularity indices $\varkappa_1 \ldots . \varkappa_n$, besides being stationary to the degree $\varkappa_1$, a point of contact with the tangent of the order $\varkappa_1 + \varkappa_2 + 1$, and with the osculating plane of the order $\varkappa_1 + \varkappa_2 + \varkappa_3 + 1$, is a point of contact of the order $\varkappa_1 + \varkappa_2 + \varkappa_3 + \varkappa_4 + 1$ with the osculating S_3, etc.

Again, if the system of osculating planes be taken as the elements of a curve of double curvature, one of these may be singular. This singularity, like the singularity of a point, is capable of definition by three indices. The first of these represents the number of consecutive coefficients congruent with the singular element in the tangential equation of the curve for its neighborhood, and the other two have the same sort of correspondence to $\varkappa_2$ and $\varkappa_3$.

In the paper already referred to, it was shown that the singularity indices of a plane element of a curve of double curvature corresponding to a point element of the singularity indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ are $\varkappa_3$, $\varkappa_2$, $\varkappa_1$. I wish now to prove, for a curve of any degree of curvature, $n-1$, that *$\varkappa_n$, $\varkappa_{n-1}$, $\ldots . \varkappa_1$ are the singularity indices of the osculating S_{n-1} which corresponds to a point of the singularity indices $\varkappa_1$, $\varkappa_2$, $\ldots . \varkappa_n$.*

Let the equation $P_t = A_0 + A_1 t + A_2 t^2 + \text{etc.}$, referred hitherto to a system of coordinates consisting of any set of $n+1$ linearly independent points, be transformed to the system $E, e_1, e_2, \ldots . e_n$, of which E is A_0 and $e_1, e_2, \ldots . e_n$ are unit vectors each at right angles to all the rest and so directed that Ee_1 is the tangent line to the curve at A_0, Ee_1e_2 the tangent S_2, $\ldots . Ee_1e_2 \ldots . e_{n-1}$ the tangent S_{n-1}. It will become

$$P_t = E + (\alpha t^{\alpha_1} + \text{etc.})\, e_1 + (\beta t^{\alpha_2} + \text{etc.})\, e_2 + \text{etc.} \ldots . + (\nu t^{\alpha_n} + \text{etc.})\, e_n,$$

where

$$\alpha_1 = \varkappa_1 + 1,$$
$$\alpha_2 = \varkappa_1 + \varkappa_2 + 2,$$
$$\ldots\ldots\ldots\ldots$$
$$\alpha_n = \sum_1^n \varkappa_i + n.$$

The thing to be especially noticed is that the sufficient and necessary conditions of point singularities of the various classes do, when the curve is referred

to the system of coordinates described, find full expression in $\alpha_1, \alpha_2, \ldots . \alpha_n$, the exponents of the lowest powers of t in the coefficients of $e_1, e_2, \ldots . e_n$ respectively, in this equation; $\alpha_1 - 1$ being the index of singularity of class 1, $\alpha_2 - \alpha_1 - 1$ the index of singularity of class 2, etc.

To demonstrate our theorem, therefore, it is only necessary to show that in the tangential equation of the curve for the neighborhood of the osculating S_{n-1} at A_0 the successive exponents corresponding to $0, \alpha_1, \alpha_2$, etc., are $0, \alpha_n - \alpha_{n-1}$, $\alpha_n - \alpha_{n-2}, \ldots . \alpha_n - \alpha_1, \alpha_n$; since, by a simple retracing of the above reasoning, the index of singularity of class 1 of this osculating S_{n-1} (we may call it ε_0) would then be $\alpha_n - \alpha_{n-1} - 1$, *i. e.* x_n; of class 2, $(\alpha_n - \alpha_{n-2}) - (\alpha_n - \alpha_{n-1}) - 1$, *i. e.* $x_{n-1} \ldots .$; finally, of class n, $\alpha_n - (\alpha_n - \alpha_1) - 1$, *i. e.* x_1.

But the tangential equation of the curve for the neighborhood of ε_0 is the product $P_t . \frac{d^{\alpha_1} P_t}{dt^{\alpha_1}} . \frac{d^{\alpha_2} P_t}{dt^{\alpha_2}} \ldots . \frac{d^{\alpha_{n-1}} P_t}{dt^{\alpha_{n-1}}}$, as is easily inferred from the definition of a tangent S_{n-1}, by getting the limit, as dt approaches 0, of the product $P_{t+\rho_1 dt} . P_{t+\rho_2 dt} \ldots . P_{t+\rho_n dt}$. Expanding, we have

$$\varepsilon_t = P_t . \frac{d^{\alpha_1} P_t}{dt^{\alpha_1}} . \frac{d^{\alpha_2} P_t}{dt^{\alpha_2}} \ldots . \frac{d^{\alpha_{n-1}} P_t}{dt^{\alpha_{n-1}}}$$

$$= f_{(n)}(t) E e_1 e_2 \ldots . e_{n-1} + f_{(n-1)}(t) E e_1 e_1 \ldots . e_{n-2} e_n + \ldots . f_{(0)}(t) e_1 e_2 \ldots . e_n,$$

any coefficient $f_{(n-j)}(t)$ being the determinant of the coefficients of

$$E, e_1, e_2, \ldots . e_{n-j-1}, e_{n-j+1}, \ldots . e_n \text{ in } P_t, \frac{d^{\alpha_1} P_t}{dt^{\alpha_1}}, \text{ etc.}$$

As we are concerned only to determine what is the lowest power of t in each of these determinants, we write, instead of the elements themselves, the exponent of the lowest power of t in each element, and, after a simple reduction, have for $f_{(n-j)}(t)$,

$$\left|\begin{array}{cccccc|ccc}
0 & \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \ldots . & \alpha_{j-1} - \alpha_1 & & \alpha_{j+1} - \alpha_1 & \ldots . & \alpha_n - \alpha_1 \\
0 & 0 & \alpha_3 - \alpha_2 & \ldots . & \alpha_{j-1} - \alpha_2 & & \alpha_{j+1} - \alpha_2 & \ldots . & \alpha_n - \alpha_2 \\
0 & 0 & 0 & \ldots . & \alpha_{j-1} - \alpha_3 & & \alpha_{j+1} - \alpha_3 & \ldots . & \alpha_n - \alpha_3 \\
. & . & . & \ldots . & . & & . & \ldots . & . \\
. & . & . & \ldots . & . & & . & \ldots . & . \\
. & . & . & \ldots . & \alpha_{j-1} - \alpha_{j-2} & & & & \\
0 & 0 & 0 & \ldots . & 0 & & \alpha_{j+1} - \alpha_{j-1} & \ldots . & \alpha_n - \alpha_{j-1} \\
\hline
0 & 0 & 0 & \ldots . & 0 & & \alpha_{j+1} - \alpha_j & \ldots . & \alpha_n - \alpha_j \\
0 & 0 & 0 & \ldots . & 0 & & 0 & \ldots . & \alpha_n - \alpha_{j+1} \\
. & . & . & \ldots . & . & & . & \ldots . & . \\
. & . & . & \ldots . & . & & . & \ldots . & . \\
0 & 0 & 0 & \ldots . & . & & 0 & \ldots . & \alpha_n - \alpha_{n-1}
\end{array}\right| .$$

The lowest power of t in $f_{(n-j)}(t)$ is then the smallest of the sums, each of n elements, of the above array so taken that no two are in the same row or column; and this smallest sum is $a_n - a_j$, as the following considerations show.

It is indeed necessary only to demonstrate that $a_n - a_j$ is the smallest of the sums built in the manner described out of the elements of the minor $\ulcorner$; for then $a_n - a_j$, plus the smallest sum of $\lrcorner$, is $a_n - a_j$, while complete sums which do not involve the sums of $\llcorner$ must involve one of the numbers $a_n - a_1, a_n - a_2, \ldots,$ $a_n - a_{j-1}$, and any of these is itself greater than $a_n - a_j$. For convenience, we substitute a, b, c, etc., for a_j, a_{j+1}, etc., and write $\ulcorner$ in the form

$$\left|\begin{array}{ccccc} b-a & c-a & d-a & e-a \ldots\ldots & m-a \\ 0 & c-b & d-b & e-b \ldots\ldots & m-b \\ 0 & 0 & d-c & e-c \ldots\ldots & m-c \\ 0 & 0 & 0 & e-d \ldots\ldots & m-d \\ \cdot & \cdot & \cdot & \cdot \ldots\ldots & \cdot \\ \cdot & \cdot & \cdot & \cdot \ldots\ldots & \cdot \\ 0 & 0 & 0 & 0 \ldots\ldots & m-l \end{array}\right|.$$

It is to be noted that any element is less than that following it in the same row, greater than that following it in the same column. The least sum in

$$\left|\begin{array}{ccc} b-a & c-a & d-a \\ 0 & c-b & d-b \\ 0 & 0 & d-c \end{array}\right|$$

is obviously $d-a$. But from this it follows that the least sum in

$$\left|\begin{array}{cccc} b-a & c-a & d-a & e-a \\ 0 & c-b & d-b & e-b \\ 0 & 0 & d-c & e-c \\ 0 & 0 & 0 & e-d \end{array}\right|$$

is $e-a$; for $e-d+$ the least sum in the minor of $e-d$ is the least sum in which $e-d$ occurs, and $e-d+d-a=e-a$; $e-c+$ the least sum in the minor got by striking out the last two rows and columns of the determinant is the least in which $e-c$ occurs, since this minor involves lower elements than any other that can be constructed out of the first two rows of the determinant, and $e-c+c-a=e-a$; and $e-b+b-a$, the minor got by canceling the last three rows and columns of the determinant, is the least in which

$e - b$ occurs, and is again $e - a$. And, by the same method of proof, it follows that since the theorem holds for determinants of the orders $1 - 4$, it holds for one of the order 5, etc.

The exponent of the lowest power of t in the coefficient of any term, $Ee_1e_2 \dots . e_{j-1} . e_{j+1} \dots . e_n$, is therefore $\alpha_n - \alpha_j$, as was to be proved, that of $e_1e_2 \dots . e_n$ in particular being α_n. Calling $Ee_1e_2 \dots . e_{j-1}e_{j+1} \dots . e_n$, ε_j, we have, as the tangential equation of the curve,

$$\varepsilon_t = \varepsilon_0 + (\alpha' t^{\alpha_n - \alpha_{n-1}} + \text{etc.})\,\varepsilon_1 + (\beta' t^{\alpha_n - \alpha_{n-2}} + \text{etc.})\,\varepsilon_2 + \text{etc.}, \dots . (\nu' t^{\alpha_n} + \text{etc.})\,\varepsilon_n .$$

PRINCETON, August 10th, 1886.

A Note on Pencils of Conics.

BY HENRY DALLAS THOMPSON.

Let the eight points in which a conic intersects a quartic be divided into two groups of four, and a conic be passed through each group: the two residual (four-point) groups lie on a conic.

Cayley's Theorem: A curve, $f_n = 0$, is cut by another, $\phi_m = 0$. The curve $\Phi_M = 0$ (where $M = m + n - \mu$, with $\mu = 1, 2, 3 \ldots . n$ or m), which passes through $mn - \frac{1}{2}(\mu - 1)(\mu - 2)$ of the points of intersection of $f_n = 0$ and $\phi_m = 0$, will also pass through the remaining $\frac{1}{2}(\mu - 1)(\mu - 2)$ points, *unless* these determined points be on a curve of the $(\mu - 3)^{\text{th}}$ order.*

Let $n = m = M = \mu = 4$; then, if a quartic be cut by a quartic, the quartic through thirteen of the points of intersection passes through the remaining three points.

Let $\phi_4 = 0$ and $\Phi_4 = 0$ each degenerate into a pair of conics. Then, if through a quartic (f) a conic (Φ) be passed, and through each of the two four-point groups of the points of intersection a conic (ϕ) be passed, the remaining eight points of intersection of the two latter conics (ϕ) with the quartic lie on a conic (Φ). For, through the remaining four points of the one conic and one of the remaining points of the other passes a conic (Φ). This, with the first conic (Φ), will pass through thirteen of the points of intersection of $f = 0$ and $\phi = 0$; hence, also, through the remaining three points, *unless* these three points lie on a

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* Cayley's Theorem: Cambridge Math. Journal, III, 211. Cayley overlooked the exceptional case mentioned. This case was discussed and completely disposed of by Bacharach: "Ueber Schnittpunkt-system Algebraischer Curven." Inaugural Dissertation, Erlangen, 1881. Also, "Ueber den Cayley'-schen Schnittpunktsatz." Math. Ann. XXVI, p. 275.

straight line ($C_{\mu-3}$); but this is impossible, as these three points lie on a conic (ϕ).*

(If one of the ϕ's degenerates into a pair of straight lines, the four points of one of the first point-groups cannot be taken except two and two on each line. Hence the four points of the corresponding latter point-group must also lie two and two on each line, so that the theorem holds where the conics degenerate.)

The following are particular cases:

If any two conics have each double contact with a third conic, then any two of the pencil of conics through the four points of contact intersect the two tangent conics in eight points which lie on a conic.

If any two conics have each double contact with a third conic, then the eight tangents, which any two of the pencil of conics touching the four tangents of contact have in common with the first pair, touch a conic.

For the two tangent conics may be taken as the original quartic (f), and the two conics of the pencil are the cutting curve (ϕ), and through eight of the points of intersection (four of contact) passes the other conic (Φ),† leaving the remaining eight points of intersection to lie on a conic (Φ).

Let four tangents be drawn to any conic, and, at the four points of contact, let a pair of doubly tangent conics be drawn; then the eight points of intersection of these two tangent conics with the tangent lines lie on a conic.

Take any four points on a conic (or pair of lines). Let a pair of conics, tangent at these four points, be drawn; then the eight intersections of this pair of conics with a pair tangent at these four points, grouped differently, lie on a conic.

If, at each of two points on a conic, a conic be drawn with contact of the third order, and if a pencil of conics be taken touching the original conic in both of the points, then any pair of this pencil will intersect the conics with third order of contact in eight points lying on a conic.

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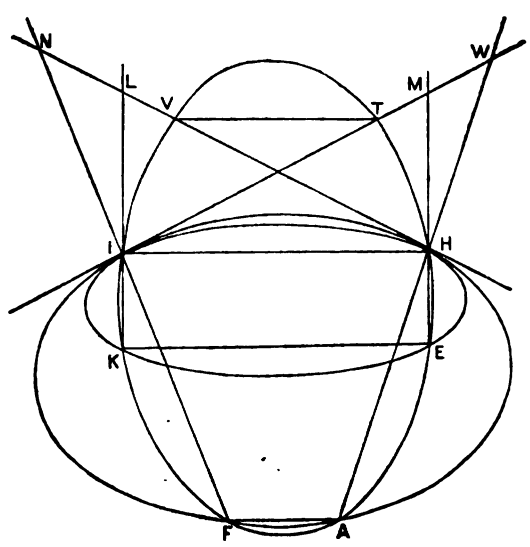
* This may be proved directly from the Riemann-Roch Theorem, $q = Q - N + \tau$, where q is the mobility of point-group, Q number of points in the group, $N = \frac{1}{2}(n-1)(n-2)$, and τ is the number of linearly independent curves of the $(n-3)^{\text{th}}$ order through the group. In the case under consideration, $N = 3$, let $q = 0$ and assume $\tau = 0$; hence $Q = 3$. If $\tau = 1$, $Q = 2$.

† It is obvious that the determining conic Φ does actually pass through eight of the points of intersection of f and ϕ. For, take the pair of conics of the pencil as each through a different four-point group on the third conic. The case considered is that when each point of one group becomes infinitely near to one of the other group.

Let any four points on a conic lie two and two on a pair of lines, and at the two points of intersection of one of these lines let a conic be taken doubly tangent to the original conic, and let another conic be taken through the four points; the last two conics will intersect again in two points collinear with the point of intersection of the two lines.

Take four tangents to any conic. Let a conic be taken tangent to the conic at two of the points of contact of tangent lines; let another conic be taken touching the four tangents; the last two conics will have two other common tangents which are concurrent with the join of the point of intersection of the two tangents taken and the point of intersection of the other two tangent lines.

For, in the figure, let IH and AF be the two lines taken; IHE the tangent conic, and $IHAFK$ the last conic. The tangents at the points I and H to the original conic intersect the last conic in V and T. The two last conics intersect in K and E. Join H and E, H and A, E and K, F and I, K and I, V and T.



Then, by Pascal's Theorem:*

a). The polygon in the tangent conic being IK, KE, EH, tangent at H, HI, tangent at I, the join of L and M, IH and KE are concurrent.

b). The polygon in the conic $IHAFK$ being VT, TI, IK, KE, EH and HV, the join of L and M, VT and KE are concurrent.

* *Vide* Cremona: Projective Geometry, Chap. XVI.

c). The polygon in the original conic being IF, FA, AH, tangent at H, HI, and tangent at I, the join of N and W, IH and FA are concurrent.

d). The polygon in the conic $IHAFK$ being VT, TI, IF, FA, AH and HV, the join of N and W, VT and AF are concurrent.

From *a*) and *b*) VT, IH and KE are concurrent. From *c*) and *d*) VT, IH and FA are concurrent. Hence IH, KE and FA are concurrent.

If, at the two points of intersection of the other line (FA), a doubly tangent conic be taken and another conic be passed through the four points, the eight points of intersection of the tangent conics with the two of the pencil (lying on a conic) lie two and two on four lines concurrent with the two joins taken of the original four points.

Any two of the pencil of conics through four points cut the tangents at these points to a third conic of the pencil in eight points, which lie on a conic having the same Conjugate Triad with the original pencil of conics (i. e. the vertices of the quadrilateral through the four points).

Any two of a pencil (line) of conics touching four lines have eight tangents, which pass through the four points of contact of another one of the conics of the pencil with the four lines. These eight lines touch a conic having the same line Conjugate Triad with the original pencil.

For if successive pairs of the tangents at the points I, H, A and F (in the preceding figure) be taken as the tangent conic, the figure will be symmetrical. Hence the join of *any* two of the eight points which are taken on the same tangent conic will pass through one of the vertices of the quadrilateral $IHAF$.

The eight-point conic can degenerate only when two points from both of the cutting conics are collinear with one of the vertices of the quadrilateral through the four points of contact. Hence,

The conic through the eight points of intersection of the tangents to, with two pairs of the joins of, four points on a conic cannot degenerate.

In this latter case the eight points lie on a conic by the first extension of Pascal's theorem. For the four tangents and two pairs of joins form a polygon of eight sides in a conic, the tangents being alternate sides. Hence the external points of intersection of tangents with joins lie on a conic.

PRINCETON, April 14, 1886.

Observations on the Generating Functions of the Theory of Invariants.

BY CAPTAIN P. A. MACMAHON, R. A.

The source of a covariant is usually understood to be the literal coefficient of the highest power of x; the continuous performance of a certain operation produces the remaining coefficients in succession.

The coefficient of the highest power of y has an equal claim to be considered a source, since the remaining coefficients are derivable by the similar performance of another operation. Moreover, if $f(a_0, a_1, a_2, \ldots.)$ be the x-source of a quantic of order ρ, then will the corresponding y-source be

$$f(a_{\rho-0}, a_{\rho-1}, a_{\rho-2}, \ldots.).$$

We see that a covariant is a perfectly symmetrical form; it is thus proper to keep both ends of the covariant in view in any investigation of the nature of sources; and, in particular, this is important when laws are being enunciated in regard to the number of sources of any specified type. Consider, as usual, a quantic of order j, and, in connection therewith, algebraic forms of weight $-w$, degree $-i$ and of extent j at most; say these forms are of type (w, i, j). We may discuss x-sources or y-sources of this type, or we may have regard to intermediate coefficients, which Sylvester, in the Phil. Mag. for 1878, called differentiants, so many removes from x or y; thus the coefficient of the highest power but p of x, in a covariant, would be a differentiant, p removes from x, and $\varepsilon - p$ removes from y, where ε is the order of the covariant. In seeking the number of x-sources of type (w, i, j), we have to count the number of partitions of w into i parts none greater than j, and also the number of similar partitions of $w - 1$; subtracting the latter number from the former, a number is obtained which, if it be positive, is the number of x-sources of the type in question; but if this difference be negative, it has, as regards x-sources, no interpretation.

To each x-source corresponds a y-source which satisfies the partial differential equation

$$jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \ldots.$$

If the x-source be of type (w, i, j), the connected y-source is of type

$$(w + ji - 2w, i, j) \equiv (ji - w, i, j),$$

and there will be intermediate coefficients $ji - 2w - 1$ in number whenever this number is non-negative; these will be of types

$$\begin{array}{l}(w+1, i, j),\\ (w+2, i, j),\\ \cdots\cdots\cdots\\ \cdots\cdots\cdots\\ \cdots\cdots\cdots\\ (ji - w - 1, i, j),\end{array}$$

respectively.

Had we, *ab initio*, sought the number of y-sources of type $(ji - w, i, j)$, we should have written down the terms of this type, each attached to an arbitrary coefficient, and, by operation of the y-source annihilator, have obtained a set of equations between them in number equal to the number of terms of type $(ji - w + 1, i, j)$. Assuming the non-existence of syzygies connecting these equations, the number of y-sources of the given type would be the difference (when positive), $[ji - w; i, j] - [ji - w + 1, i, j]$, wherein $[w, i, j]$ denotes the number of partitions of type (w, i, j).

We have, as a known theorem of partitions,

$$[wij] - [w-1, i, j] = [ji - w, i, j] - [ji - w + 1, i, j],$$

from which the conclusion is that no syzygies exist between the foregoing set of equations, and that, as a fact, the number of y-sources of type $(w - 1, i, j)$ is precisely $[w - 1, i, j] - [w; i, j]$, when this number happens to be positive.

If $[w - 1, i, j] = [w, i, j]$, there exists neither an x-source of type (w, i, j) nor a y-source of type $(w - 1, i, j)$.

There may, however, exist intermediate coefficients of either type, since the dexter sides of the equations arrived at will, in general, not be zero.

We are now in a position to give a more complete statement of the Cayley-Sylvester theorem under discussion.

NEW STATEMENT OF THEOREM.

The number $$[w, i, j] - [w-1, i, j]$$
is, when positive, the number of x-sources of type (w, i, j); whilst, when
$$[w-1, i, j] - [w, i, j]$$
is positive, it denotes the number of y-sources of type
$$(w-1, i, j).$$
This is equivalent to saying that the coefficient of x^w in the ascending expansion of
$$\frac{1-x^{j+1}, \ldots . 1-x^{j+i}}{1-x^2, \ldots . 1-x^i}$$
is, when positive, the number of x-sources of type (w, i, j); whilst, when in the same expansion the coefficient of $-x^w$ is positive, it denotes the number of y-sources of type $(w-1, i, j)$.

It is somewhat remarkable that this explanation of the complete expansion of the generating function, simple and fundamental as it is, appears to have hitherto escaped the notice of writers on the subject.

It will now be clear that we may write
$$\frac{1-x^{j+1}.\,1-x^{j+2}.\,\ldots . 1-x^{j+i}}{1-x^2.\,1-x^3.\,\ldots . 1-x^i} = \Sigma A_w (x^w - x^{ji+1-w}),$$
A_w being a positive number and denoting the number of x-sources of type (w, i, j).

We may put $$x = \cosh\phi + \sinh\phi,$$
and then
$$x^s + x^{-s} = 2\cosh s\phi,$$
$$x^s - x^{-s} = 2\sinh s\phi,$$
leading to the identity
$$\frac{\sinh\frac{1}{2}(j+1)\phi \sinh\frac{1}{2}(j+2)\phi \ldots . \sinh\frac{1}{2}(j+i)\phi}{\sinh\phi \sinh\frac{3}{2}\phi \ldots . \sinh\frac{1}{2}i\phi}$$
$$= \sum A_w \sinh\frac{1}{2}(ji+1-2w)\phi;$$
or, writing $\sqrt{-1}\psi$ for ϕ, so that
$$\sinh\phi = \sqrt{-1}\sin\psi,$$

we have also

$$\frac{\sin\frac{1}{2}(j+1)\psi\,\sin\frac{1}{2}(j+2)\psi\ldots\ldots\sin\frac{1}{2}(j+i)\psi}{\sin\psi\,\sin\frac{3}{2}\psi\ldots\ldots\sin\frac{1}{2}i\psi}$$

$$=\sum A_w\sin\frac{1}{2}(ji+1-2w)\psi.$$

In a certain sense the left-hand members of these two identities are pure generating functions in the theory, since they are omni-positive in development.

A_w may be represented as the coefficient in a Fourier series, its value as a definite integral being

$$\frac{2}{\pi}\int_0^\pi\frac{\sin\frac{1}{2}(j+1)\psi\ldots\ldots\sin\frac{1}{2}(j+i)\psi}{\sin\psi\ldots\ldots\sin\frac{1}{2}i\psi}\sin\frac{1}{2}(ji+1-2w)\psi d\psi.$$

On the Transformation of Elliptic Functions.

BY PROFESSOR CAYLEY.

The algebraical theory of the Transformation of Elliptic Functions was established by Jacobi in a remarkably simple and elegant form, but it has not hitherto been developed with much completeness or success. The cases $n = 3$ and $n = 5$ are worked out very completely in the Fundamenta Nova (1829); viz., considering the equation

$$\frac{Mdy}{\sqrt{1-y^2 . 1-\lambda^2 y^2}} = \frac{dx}{\sqrt{1-x^2 . 1-k^2 x^2}},$$

($k = u^4$, $\lambda = v^4$; say this is the $Mk\lambda$- or Muv-form) Jacobi finds in the two cases respectively, a modular equation between the fourth roots u, v, say the uv-modular equation, and, as rational functions of u, v, the value of M and the values of the coefficients of the several powers of x in the numerator and denominator of the fraction which gives the value of y; but there is no attempt at a like development of the general case. I shall have occasion to speak of other researches by Jacobi, Brioschi and myself; but I will first mention that my original idea in the present memoir was to develop the following mode of treatment of the theory:

In place of the $Mk\lambda$-form, using the $\rho\alpha\beta$-form

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}}$$

(I write for greater convenience 2α, 2β in place of the α of Jacobi and Brioschi and the β of Brioschi), we can, by expanding each side in a series, integrating and reverting the resulting series for y, obtain y in the form

$$y = \rho x\,(1 + \Pi_1 x^2 + \Pi_2 x^4 + \ldots.),$$

where $\Pi_1, \Pi_2, \Pi_3, \ldots.$ denote given functions of ρ, α, β. Taking n odd and $= 2s+1$, we assume for y an expression

$$y = \frac{x(A_s + A_{s-1}x^2 \ldots. + A_1x^{2s-2} + x^{2s})}{1 + A_1x^2 \ldots. + A_{s-1}x^{2s-2} + A_sx^{2s}},$$

where the last coefficient A_s is at once seen to be $= \rho$. Comparing with the series-value $y = \rho x(1 + \Pi_1x^2 + \Pi_2x^4 + \ldots.)$, we have an infinite series of equations. The first of these is, in fact, $A_s = \rho$; the next $(s-1)$ equations give linearly $A_1, A_2, \ldots. A_{s-1}$ in terms of the coefficients Π; that is, of ρ, α, β: the two which follow serve in effect to determine ρ, β as functions of α: and then, ρ, β having these values, all the remaining equations will be satisfied identically.

The process is an eminently practical one, so far as regards the determination of the coefficients $A_1, A_2, \ldots. A_{s-1}$ as functions of ρ, α, β; it is less so, and requires eliminations more or less complicated, as regards the determination of the relations between ρ, α, β. As to this, it may be remarked that the problem is not so much the determination of the equation between ρ and α (or say the $\rho\alpha$-multiplier equation, or simply the $\rho\alpha$-equation), and of the equation between β, α (or say the $\alpha\beta$-modular equation, or simply the $\alpha\beta$-equation), as it is to determine the complete system of relations between ρ, α, β; treating these as coordinates, we have what may be called the multiplier-modular-curve, or say the MM-curve, and the relations in question are those which determine this curve.

In the absence of special exceptions, it follows from general principles that the coefficients $A_1, A_2, \ldots. A_{s-1}$, quà rational functions of ρ, α, β, must also be rational functions of α, β or of α, ρ; and I think it may be assumed that this is the case; the method, however, affords but little assistance towards thus expressing them.

In connection with the foregoing theory, I consider the solutions of the problem of transformation given by Jacobi's partial differential equation (Suite de Notices sur les Fonctions elliptiques: Crelle, t. IV (1829), pp. 185–193), and by what I call the Jacobi-Brioschi differential equations. The first and third of these were obtained by Jacobi in the memoir, De functionibus ellipticis Commentatio: Crelle, t. IV (1829), pp. 371–390 (see p. 376); but the second equation, which completes the system, was, I believe, first given by Brioschi in the second appendix to his translation of my Elliptic Functions: Trattato elementare delle Funzioni ellittiche: Milan, 1880. I had, strangely enough, overlooked the great

importance of these equations. I shall have occasion also to refer to results, and further develop the theory contained in my memoir, On the Transformation of Elliptic Functions: Phil. Trans., t. 163 (1873), pp. 397–456, and the addition thereto: Phil. Trans., t. 188 (1878), pp. 419–424.

I remark that while I have only worked out the formulæ for the cases $n=3$ and $n=5$, and a few formulæ for the case $n=7$, the memoir is intended to be a contribution to the general theory of the $\rho\alpha\beta$-transformation; I hope to be able to complete the theory for the case $n=7$.

Comparison of the $Mk\lambda$- and $\rho\alpha\beta$-Forms. The Modular and Multiplier Equations. Art. Nos. 1 *to* 12.

1. The equation

$$\frac{M d\mathrm{y}}{\sqrt{1-\mathrm{y}^2 . 1-\lambda^2\mathrm{y}^2}}=\frac{d\mathrm{x}}{\sqrt{1-\mathrm{x}^2 . 1-k^2\mathrm{x}^2}},$$

if we write therein $\quad \mathrm{x}=\frac{x}{\sqrt{k}},\ =\frac{x}{u^2};\ \mathrm{y}=\frac{y}{\sqrt{\lambda}},\ =\frac{y}{v^2},$

becomes $\quad \frac{M dy}{v^2\sqrt{1-(v^4+v^{-4})y^2+y^4}}=\frac{dx}{u^2\sqrt{1-(u^4+u^{-4})x^2+x^4}};$

viz., this is $\quad \frac{dy}{\sqrt{1-2\beta y^2+y^4}}=\frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}},$

if only $\quad 2\alpha=u^4+\frac{1}{u^4},\ 2\beta=v^4+\frac{1}{v^4},\ \rho=\frac{v^2}{u^2M}.$

2. We have a uv-modular equation, and, as shown in my Transformation Memoir, p. 450, this may be converted into a u^4v^4-modular equation; in particular, $n=3$, the equation is

$$y^4+6x^2y^2+x^4-4xy\,(4x^2y^2-3x^2-3y^2+4)=0,$$

where x, y denote u^4, v^4 respectively; say the equation is

$$F(x,\,y),\ =x^4+x^3(-16y^3+12y)+x^2(6y^2)+x(12y^3-16y)+y^4,\ =0.$$

From the equation $F(x,y)=0$, we derive

$$x^{-2}F(x,\,y)\,.\,x^{-2}F(x,\,y^{-1})=0;$$

say this is

$$(Ax^2+Bx+C+Dx^{-1}+Ex^{-2})(A'x^2+B'x+C'+D'x^{-1}+E'x^{-2})=0,$$

viz., the equation is

$$AA'x^4 + (AB' + A'B)\,x^3 \ldots . + EE'x^{-4} = 0\,,$$

where, by reason of the symmetry of $F(x,\, y)$, the coefficients AA', EE' of x^4, x^{-4}, those of x^3, x^{-3}, etc., have equal values; the form thus is

$$\mathfrak{A}\,(x^4 + x^{-4}) + \mathfrak{B}\,(x^3 + x^{-3}) + \mathfrak{C}\,(x^2 + x^{-2}) + \mathfrak{D}\,(x + x^{-1}) + \mathfrak{E} = 0\,,$$

where $x^4 + x^{-4}$, $x^3 + x^{-3}$, $x^2 + x^{-2}$, are given functions of $x + x^{-1}$, $= 2\alpha$; viz., we have

$$\begin{aligned} x + x^{-1} &= 2\alpha\,, \\ x^2 + x^{-2} &= 4\alpha^2 - 2\,, \\ x^3 + x^{-3} &= 8\alpha^3 - 6\alpha\,, \\ x^4 + x^{-4} &= 16\alpha^4 - 16\alpha^2 + 2\,, \end{aligned}$$

and the coefficients $\mathfrak{A}$, $\mathfrak{B}$, ... are in like manner expressible as functions of $y + y^{-1}$, $= 2\beta$; thus we have $\mathfrak{A} = 1$, $\mathfrak{B} = AB' + A'B$

$$= -16\,(y^3 + y^{-3}) + 12\,(y + y^{-1}),\ = -16\,(8\beta^3 - 6\beta) + 12\,.\,2\beta\,;$$

or, finally, $\mathfrak{B} = -128\beta^3 + 120\beta$; and so for the other coefficients. The numerical coefficients contain, all of them, the factor 16; and, throwing this out, we obtain ($n = 3$) the $\alpha\beta$-modular equation in the form

	α^4	α^3	α^2	α	1
β^4					$+1$
β^3		-64		$+60$	
β^2			-186		$+192$
β		$+60$		-64	
1	$+1$		$+192$		-192
	$+1$	-4	$+6$	-4	$+1$

$= 0\,,$

where observe that the form is symmetrical as regards α, β; and, further, that the sums of the numerical coefficients in the lines or columns are the binomial coefficients 1, -4, $+6$, -4, $+1$. Observe, further, that the sums in the direction of the sinister diagonal are -64, -64, $+320$, -192; viz., dividing

by -64, it thus appears that, writing $\beta=\alpha$, the equation becomes

$$\alpha^6+\alpha^4-5\alpha^2+3=0;$$

that is, $(\alpha^2-1)^2(\alpha^2+3)=0$.

Again, writing $\beta=-\alpha$, then dividing by 16, the equation becomes

$$4\alpha^6-19\alpha^4+28\alpha^2-12=0;$$

that is, $(4\alpha^2-3)(\alpha^2-2)^2=0$.

3. So also, $n=5$, we have the u^4v^4-modular equation in the form

$$\left.\begin{array}{l} x^6+655x^4y^2+655x^2y^4+y^6-640x^3y^3-640x^4y^4 \\ \quad +\,xy\,(-256+320x^2+320y^2-70x^4-660x^2y^2-70y^4 \\ \qquad +\,320x^4y^2+320x^2y^4-256x^4y^4) \end{array}\right\}=0;$$

and in precisely the same manner we obtain the $\alpha\beta$-modular equation; viz. (casting out a factor 64), this is

	β^6	β^5	β^4	β^3	β^2	β	1
α^6							$+1$
α^5		-4096		$+6400$		-2310	
α^4			$+69120$		-172785		$+103680$
α^3		$+6400$		-133140		$+126720$	
α^2			-172785		$+276480$		-103680
α		-2310		$+126720$		-124416	
1	$+1$		$+103680$		-103680		
	$+1$	-6	$+15$	-20	$+15$	-6	$+1$

$=0,$

where the form is symmetrical as regards α, β; the sums of the numerical coefficients in the lines or columns are $1, -6, +15, -20, +15, -6, +1$. The sums in the direction of the sinister diagonal all divide by -4096; viz., throwing out this factor, we have for $\beta=\alpha$ the equation

$$\alpha^{10}-20\alpha^8+118\alpha^6-180\alpha^4+81\alpha^2=0;$$

that is, $\alpha^2(\alpha^2-1)^2(\alpha^2-9)^2=0$.

If $\beta=-\alpha$, the coefficients divide by 64; and throwing out this factor, the equation is $64\alpha^{10}+880\alpha^8-3247\alpha^6+3600\alpha^4-1296\alpha^2=0$;

that is, $\alpha^2(\alpha^4+16\alpha^2-16)(8\alpha^2-9)^2=0$.

4. We have a Mu-multiplier equation of the form $F\left(\frac{1}{M},\ 2u^8-1\right)=0$ (see Memoir, pp. 420–422), but we cannot, by the preceding formulæ, deduce thence a $\rho\alpha$-multiplier equation; in fact, writing therein $\frac{1}{M}=\frac{u^2\rho}{v^2}$, the resulting equation is $F\left(\frac{u^2\rho}{v^2},\ 1-2u^8\right)=0$, which is a $\rho\alpha$-multiplier equation only on the assumption that $1-2u^8$, u^2 and v^2 are therein regarded as given functions of α. But it is very remarkable that the $\rho\alpha$-equation in fact is $F(\rho,\alpha)=0$. To prove this, assume that the equation

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}}=\frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}}$$

has a $\rho\alpha$-multiplier equation $F(\rho,\alpha)=0$. Starting from the equation

$$\frac{M\mathrm{d}\mathrm{y}}{\sqrt{1-\mathrm{y}^2.1-\lambda^2\mathrm{y}^2}}=\frac{\mathrm{d}\mathrm{x}}{\sqrt{1-\mathrm{x}^2.1-k^2\mathrm{x}^2}},$$

we may, by effecting on each side a quadric transformation, convert this into

$$\frac{dy}{\sqrt{1-2(2v^8-1)y^2+y^4}}=\frac{M^{-1}dx}{\sqrt{1-2(2u^8-1)x^2+x^4}};$$

and this being so, we have, between M^{-1} and $2u^8-1$, the relation

$$F\left(\frac{1}{M},\ 2u^8-1\right)=0;$$

or, conversely, if this be the form of the Mu-multiplier equation, then the $\rho\alpha$-multiplier equation is $F(\rho,\alpha)=0$.

5. The quadric transformations are

$$x=\frac{\sqrt{1-\mathrm{x}^2}}{\mathrm{x}\sqrt{1-k^2\mathrm{x}^2}},\ y=\frac{\sqrt{1-\mathrm{y}^2}}{\mathrm{y}\sqrt{1-\lambda^2\mathrm{y}^2}}.$$

We have then only to show that

$$\frac{dx}{\sqrt{1-2(2u^8-1)x^2+x^4}}=\frac{d\mathrm{x}}{\sqrt{1-\mathrm{x}^2.1-k^2\mathrm{x}^2}};$$

for then, in like manner,

$$\frac{dy}{\sqrt{1-2(2v^8-1)y^2+y^4}}=\frac{d\mathrm{y}}{\sqrt{1-\mathrm{y}^2.1-\lambda^2\mathrm{y}^2}},$$

and we pass from the assumed differential relation between x, y to the above-mentioned differential equation between x, y.

6. For the quadric transformation between x, x, write

$$\theta^{\frac{1}{2}} = k - ik',\ \theta^{-\frac{1}{2}} = k + ik'$$

$\left(\text{whence also } \theta = \dfrac{k-ik'}{k+ik'}\right)$, and therefore

$$\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}} = 2k,\ \theta + \theta^{-1} = 2k^2 - 2k'^2 = 2(2k^2 - 1),\ = 2(2u^8 - 1);$$

we have

$$1 - \theta x^2 = 1 - \theta\frac{1-\mathrm{x}^2}{\mathrm{x}^2(1-k^2\mathrm{x}^2)},\ = \frac{1}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}\ \{-\theta + (\theta+1)\,\mathrm{x}^2 - k^2\mathrm{x}^4\},$$

$$= \frac{-\theta}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}\,(1 - \theta^{-\frac{1}{2}}k\mathrm{x}^2)^2;$$

and similarly,

$$1 - \theta^{-1}x^2 = \frac{-\theta^{-1}}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}\,(1 - \theta^{\frac{1}{2}}\,k\mathrm{x}^2)^2.$$

Consequently,

$$(1-\theta x^2)(1-\theta^{-1}x^2) = 1 - 2(2u^8-1)x^2 + x^4 = \frac{1}{\mathrm{x}^4(1-k^2\mathrm{x}^2)^2}\quad(1 - 2k^2\mathrm{x}^2 + k^2\mathrm{x}^4)^2;$$

or say

$$\sqrt{1 - 2(2u^8 - 1)\,x^2 + x^4} = \frac{1}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}\quad(1 - 2k^2\mathrm{x}^2 + k^2\mathrm{x}^4).$$

Moreover,

$$dx = \frac{d\mathrm{x}}{\mathrm{x}^2(1-\mathrm{x}^2)^{\frac{1}{2}}(1-k^2\mathrm{x}^2)^{\frac{3}{2}}}\,(1 - 2k^2\mathrm{x}^2 + k^2\mathrm{x}^4),$$

and thence the required equation

$$\frac{dx}{\sqrt{1 - 2(1-2u^8)\,x^2 + x^4}} = \frac{d\mathrm{x}}{\sqrt{1-\mathrm{x}^2 . 1 - k^2\mathrm{x}^2}};$$

this completes the proof.

7. Thus, referring to the Mu-equations given in the place referred to, we obtain the following $\rho\alpha$-multiplier equations:

$n = 3$, $\rho^4 - 6\rho^2 - 8\alpha\rho - 3 = 0$.

This may be written in the forms

$$\begin{aligned} 8\alpha\rho &= \rho^4 - 6\rho^2 - 3, \\ 8(\alpha+1)\,\rho &= (\rho-1)^3(\rho+3), \\ 8(\alpha-1)\,\rho &= (\rho+1)^3(\rho-3). \end{aligned}$$

$n = 5$, $\rho^6 - 10\rho^5 + 35\rho^4 - 60\rho^3 + 55\rho^2 + (38 - 64\alpha^2)\,\rho + 5 = 0$.

This may be written in the two forms

$$\begin{aligned} 64\alpha^2\rho &= (\rho^2 - 4\rho - 1)^2(\rho^2 - 2\rho + 5) \text{ and} \\ 64\,(\alpha^2 - 1)\,\rho &= (\rho - 1)^5(\rho - 5). \end{aligned}$$

$$n = 7,\ \rho^8 - 28\rho^6 - 112\alpha\rho^5 - 210\rho^4 - 224\alpha\rho^3 + (-1484 + 1344\alpha^2)\,\rho^2 + (-560\alpha + 512\alpha^3)\,\rho + 7 = 0.$$

8. The relation between ρ and β, or say the $\rho\beta$-multiplier equation, may be obtained by a known property of elliptic functions; viz., writing $\rho\sigma = \pm n$ (sign is $-$ for $n=3$, $n=7$, or generally for any prime value $4p+3$, and it is $+$ for $n=5$ and generally for any prime value $=4p+1$), then we have between σ, β the same relation as between ρ, α. Thus, $n=3$, $\sigma = -\dfrac{3}{\rho}$, for ρ, α writing σ, β, the equation is $\sigma^4 - 6\sigma^2 - 8\beta\sigma - 3 = 0$; or, as this may be written,

$$\rho^4 + 8\beta\rho^3 + 18\rho^2 - 27 = 0;$$

and so for the other cases; but it is perhaps more convenient to retain the σ; thus, $n=5\left(\sigma = \dfrac{5}{\rho}\right)$, we have

$$\sigma^6 - 10\sigma^5 + 35\sigma^4 - 60\sigma^3 + 55\sigma^2 + (38 - 64\beta^2)\sigma + 5 = 0.$$

9. We are hence able to express β as a rational function of ρ, α. We, in fact, have

$$8\alpha = \frac{1}{\sqrt{\rho}}(\rho^2 - 4\rho - 1)\sqrt{\rho^2 - 2\rho + 5}, \quad 8\beta = -\frac{1}{\sqrt{\sigma}}(\sigma^2 - 4\sigma - 1)\sqrt{\sigma^2 - 2\sigma + 5}$$

(the signs must be opposite), and then for σ, substituting its value $= \dfrac{\rho}{5}$ and observing that $\sigma^2 - 2\sigma + 5$ is thus $= \dfrac{5}{\rho^2}(\rho^2 - 2\rho + 5)$, we find

$$\frac{\beta}{\alpha} = \frac{\rho^2 + 20\rho - 25}{\rho^2(\rho^2 - 4\rho - 1)},$$

which is the required formula.

Observe that for $\rho = \sigma = \sqrt{5}$, the formulæ with the sign $-$, as above, give $\beta = -\alpha$, whereas with the sign $+$ they would have given $\beta = \alpha$. For the value in question, $\rho = \sqrt{5}$, the equation $64\alpha^2 = \dfrac{1}{\rho}(\rho^2 - 4\rho - 1)^2(\rho^2 - 2\rho + 5)$, gives

$$64\alpha^2 = \frac{1}{\sqrt{5}}\,16\,(1 - \sqrt{5})^2(10 - 2\sqrt{5});$$

that is, $\alpha^2 = \dfrac{1}{\sqrt{5}}(3 - \sqrt{5})(5 - \sqrt{5}), = (3 - \sqrt{5})(\sqrt{5} - 1)$; that is, $\alpha^2 = -8 + 4\sqrt{5}$, or $\alpha^4 + 16\alpha^2 - 16 = 0$; it appears, *ante* No. 3, that this value belongs to the case $\beta = -\alpha$ and not to $\beta = \alpha$.

10. But there is another way of arriving at a formula containing β. Starting from Jacobi's equation

$$nM^2 = \frac{\lambda\lambda'^2}{kk'^2}\cdot\frac{dk}{d\lambda},$$

From the equations

$$64(\alpha^2-1)=\frac{1}{\rho}(\rho-1)^5(\rho-5),\ \text{ and }\ 8\alpha=\frac{1}{\sqrt{\rho}}(\rho^2-4\rho-1)\sqrt{\rho^2+2\rho-5}$$

we have

$$\frac{128\alpha d\alpha}{\alpha^2-1}=\frac{5(\rho^2-4\rho-1)\,d\rho}{\rho(\rho-1)(\rho-5)},\ \text{ and thence }\ \frac{16d\alpha}{\alpha^2-1}=\frac{5d\rho}{(\rho-1)(\rho-5)\sqrt{\rho(\rho^2-2\rho+5)}}.$$

Similarly, observing the — sign of 8β, $\dfrac{16d\beta}{\beta^2-1}=\dfrac{-5d\sigma}{(\sigma-1)(\sigma-5)\sqrt{\sigma(\sigma^2-2\sigma+5)}}$, whence, substituting for σ its value $=\dfrac{5}{\rho}$, we have

$$\frac{16d\beta}{\beta^2-1}=\frac{\rho^2 d\rho}{(\rho-1)(\rho-5)\sqrt{\rho(\rho^2-2\rho+5)}},=\frac{\rho^2}{5}\cdot\frac{16d\alpha}{\alpha^2-1},$$

which is right.

Connection of the $Mk\lambda$- and $\rho\alpha\beta$-Theories. Order of Modular Equation. Art. Nos. 13 *to* 18.

13. In the Transformation Memoir, starting from the equation

$$\frac{1-\mathrm{y}}{1+\mathrm{y}}=\frac{1-\mathrm{x}}{1+\mathrm{x}}\left(\frac{P-Q\mathrm{x}}{P+Q\mathrm{x}}\right)^2,$$

I sought to determine the coefficients of P, Q by the consideration that the relation between x, y remains unaltered when x, y are changed into $\dfrac{1}{k\mathrm{x}}$, $\dfrac{1}{\lambda\mathrm{y}}$ respectively. This comes to saying that when for x, y we write $\dfrac{x}{u^2}$, $\dfrac{y}{v^2}$ respectively, the relation between x, y presents itself in the form

$$y=\frac{x(A_s+A_{s-1}x^2\ldots+A_0x^{2s})}{A_0+A_1x^2\ldots+A_sx^{2s}}$$

$\left(s=\dfrac{1}{2}(n-1),\text{ as before}\right)$. For instance, $(n=7)$, $P=\alpha+\gamma\mathrm{x}^2$, $Q=\beta+\delta\mathrm{x}^2$.

If, solving for y, we then for x, y write $\dfrac{x}{u^2}$, $\dfrac{y}{v^2}$, we find

$$y=\frac{v^2u^{-2}x\{(\alpha^2+2\alpha\beta)+(2\alpha\gamma+\beta^2+2\alpha\delta+2\beta\gamma)x^2u^{-4}+(\gamma^2+2\beta\delta+2\gamma\delta)x^4u^{-8}+\delta^2x^6u^{-12}\}}{\alpha^2+(2\alpha\gamma+\beta^2+2\alpha\beta)x^2u^{-4}+(\gamma^2+2\beta\delta+2\alpha\delta+2\beta\gamma)x^4u^{-8}+(\delta^2+2\gamma\delta)x^6u^{-12}};$$

and comparing this with

$$y=\frac{x(A_3+A_2x^2+A_1x^4+A_0x^6)}{A_0+A_1x^2+A_2x^4+A_3x^6},$$

we have for each of the coefficients A two different expressions, and equating these and making a slight change of form, we obtain the relations between u, v, α, β, γ, δ used in the Memoir. Thus,

$$A_0 = a^2 = v^2 u^{-14} \delta^2,\ A_1 = v^2 u^{-10} (\gamma^2 + 2\beta\delta + 2\gamma\delta) = u^{-4}(2a\gamma + \beta^2 + 2a\beta), \text{ etc.};$$

in the Memoir, $k\,(= u^4)$ is used instead of u, and $\Omega\,(= v^2 u^{-2})$ instead of v, and the equations thus are

$$\begin{gathered} k^3 a^2 = \Omega \delta^2, \\ k\,(2a\gamma + 2a\beta + \beta^2) = \Omega\,(\gamma^2 + 2\gamma\delta + 2\beta\delta), \\ \gamma^2 + 2\beta\gamma + 2a\delta + 2\beta\delta = \Omega k\,(2a\gamma + 2\beta\gamma + 2a\delta + \beta^2), \\ \delta^2 + 2\gamma\delta = \Omega k^3 (a^2 + 2a\beta); \end{gathered}$$

viz., these are the equations p. 403. The idea in the present Memoir is that of considering the coefficients A in the stead of α, β, ...

14. We have here, and in general for any odd value of n, equations of the form

$$(\Omega =) \frac{U}{U'} = \frac{V}{V'} = \ldots\ldots,$$

where $U, V, \ldots, U', V', \ldots$ are quadric functions of the coefficients $\alpha, \beta, \gamma, \ldots$, and these equations serve to establish between Ω and k a relation called the Ωk-modular equation, and which is in regard to Ω of the same degree as the uv-modular equation is in regard to v. Leaving out the equation $(\Omega =)$, we have

$$\begin{vmatrix} U, & V, & W, & \ldots \\ U', & V', & W', & \ldots \end{vmatrix} = 0;$$

and to each system of values of α, β, γ, δ .... (or say of their ratios) given by these equations, there corresponds a single value of Ω; the number of values of Ω, or degree in Ω, of the Ωk-equation is thus found as $= (n + 1)\, 2^{\frac{1}{2}(n-3)}$. This is far too high; for $n = 3, 5, 7, \ldots$ the degrees are 4, 12, 32, ... ; those of the proper Ωk-equations are 4, 6, 8, ...

15. I showed, or endeavored to show, that in the case $n = 5$, the extraneous factor was $(\Omega - 1)^6$, ($\Omega - 1 = 0$, the Ωk-modular equation belonging to $n = 1$, for which the transformation is the trivial one $y = x$), and that in the case $n = 7$, the extraneous factor was $\{(\Omega, 1)^4\}^6$, ($(\Omega, 1)^4 = 0$, the Ωk-modular equation for the case $n = 3$); generally the extraneous factors seem to depend on the Ωk-functions for the values $n - 4$, $n - 8$, etc. The ground for this is

that in the assumed formula for any given value n, we may take P, Q to contain a common factor $1 \pm k\mathrm{x}^2$ $\left(\text{observe that to a factor près this is unaltered by the change x into } \frac{1}{k\mathrm{x}}, \text{ viz. it becomes } \frac{1}{k\mathrm{x}^2}(1 \pm k\mathrm{x}^2), \text{ a condition which is necessary}\right)$, and we thereby reduce the equation to

$$\frac{1-\mathrm{y}}{1+\mathrm{y}} = \frac{1-\mathrm{x}}{1+\mathrm{x}}\left(\frac{P'-Q'\mathrm{x}}{P'+Q'\mathrm{x}}\right)^2,$$

in which equation the degrees of the numerator and denominator are each diminished by 4, and the equation thus belongs to the value $n-4$.

16. I remark here that in the case of n an odd prime, the degree of the modular equation is $=n+1$; but for any other odd value the degree is $\sigma'(n), = n\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\ldots.$, where $a, b, \ldots.$ are all the unequal prime factors of n; thus, $n=a^a$, the degree is $a^a\left(1+\frac{1}{a}\right), = a^{a-1}(a+1)$. In the case of a number $n = abc\ldots.$, without any square factor, the degree is $abc\ldots.\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\ldots., = (a+1)(b+1)(c+1)\ldots.$, the sum of the factors of n. We have

$$\sigma'(n) = \text{coeff. } x^n \text{ in } \Sigma\phi(x^N),$$

where

$$\phi x = x + 3x^3 + 5x^5 + \ldots., = \frac{x(1+x^2)}{(1-x^2)^2},$$

and the summation extends to all odd values of N having no square factor; thus,

$$\begin{array}{llllllllll}
\phi(x) & = x & + 3x^3 & + 5x^5 & + 7x^7 & + 9x^9 & + 11x^{11} & + 13x^{13} & + 15x^{15} & \ldots. \\
\phi(x^3) & = & 1x^3 & & & + 3x^9 & & & + 5x^{15}, & \\
\phi(x^5) & = & & 1x^5 & & & & & + 3x^{15}, & \\
\phi(x^7) & = & & & 1x^7 & & & & & \\
\phi(x^{11}) & = & & & & & 1x^{11} & & & \\
\phi(x^{13}) & = & & & & & & 1x^{13} & & \\
\phi(x^{15}) & = & & & & & & & 1x^{15}, & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\
\hline
\Sigma\phi(x^N) & = x & + 4x^3 & + 6x^5 & + 8x^7 & + 12x^9 & + 12x^{11} & + 14x^{13} & + 24x^{15} & \ldots.
\end{array}$$

17. Supposing that the reduction is completely accounted for as above, then, to obtain the numerical relations, the numbers 1, 4, 12, 32, ... $(n+1)\,2^{\frac{1}{2}(n-3)}$ have to be expressed linearly in terms of 1, 4, 6, 8 .... $\sigma'(n)$, viz. $(n+1)2^{\frac{1}{2}(n-3)}$ as a linear function of $\sigma'(n)$, $\sigma'(n-4)$, $\sigma'(n-8)$, ..., and we have

$$
\begin{aligned}
1 &= 1, \\
4 &= 4, \\
12 &= 6 + 6.1, \\
32 &= 8 + 6.4, \\
80 &= 12 + 6.6 + 32.1, \\
192 &= 12 + 6.8 + 33.4, \\
448 &= 14 + 6.12 + 33.6 + 164.1, \\
1024 &= 24 + 6.12 + 33.8 + 166.4, \\
2304 &= 18 + 6.14 + 33.12 + 166.6 + 810.1, \\
5120 &= 20 + 6.24 + 33.12 + 166.8 + 817.4, \\
11264 &= 32 + 6.18 + 33.14 + 166.12 + 817.6 + 3768.1, \\
24576 &= 24 + 6.20 + 32.24 + 166.12 + 817.8 + 3778.1, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

but it is of course very doubtful whether these relations have any value in regard to the present theory.

18. In the same way that, by assuming a common factor, $1 + kx^2$, in the values of P, Q, we pass from the case n to the case $n-4$, so, by assuming a common factor, $1 + x^2$, in the numerator and denominator of the expression for y in terms of x and the coefficients B, we pass from the case n to the case $n-2$. Contrariwise, in the solutions given by the Jacobi-Brioschi differential equations and by the Jacobi partial differential equation, the solution for a given value of n does *not* thus contain the solution for an inferior value of n; see *post* Nos. 36 and 43.

I pass now to the theory before referred to.

The Development $y = \rho x(1 + \Pi_1 x^2 + \Pi_2 x^4 + \ldots.)$. *Art. Nos.* 19 *and* 20.

19. Starting from the equation

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}},$$

and writing for shortness

$$R_1 = \frac{1}{3}\alpha, \qquad S_1 = \frac{1}{3}\beta,$$

$$R_2 = \frac{1}{5}\left(\frac{3}{2}\alpha^2 - \frac{1}{2}\right), \qquad S_2 = \frac{1}{5}\left(\frac{3}{2}\beta^2 - \frac{1}{2}\right),$$

$$R_3 = \frac{1}{7}\left(\frac{5}{2}\alpha^3 - \frac{3}{2}\alpha\right), \qquad S_3 = \frac{1}{7}\left(\frac{5}{2}\beta^3 - \frac{3}{2}\beta\right),$$

$$R_4 = \frac{1}{9}\left(\frac{35}{8}\alpha^4 - \frac{15}{4}\alpha^2 + \frac{3}{8}\right), \qquad S_4 = \frac{1}{9}\left(\frac{35}{8}\beta^4 - \frac{15}{4}\beta^2 + \frac{3}{8}\right),$$

. . . .

. . . .

(viz., save as to the exterior factors $\frac{1}{3}$, $\frac{1}{5}$, ..., R_1, R_2, ... are the Legendrian functions of α, and S_1, S_2, ... the Legendrian functions of β), we have

$$dy\,(1 + 3S_1y^2 + 5S_2y^4 + \ldots.) = \rho dx\,(1 + 3R_1x^2 + 5R_2x^4 + \ldots.),$$

whence integrating, so that y may vanish with x, we have

$$y + S_1y^3 + S_2y^5 + \ldots. = \rho\,(x + R_1x^3 + R_2x^5 + \ldots.),$$

say this is $\qquad = u.$

20. We then have $y = u - fy$ where $fy = S_1y^3 + S_2y^5 + \ldots.$, and thence, expanding by Lagrange's theorem,

$$y = u - fu + \frac{1}{2}(f^2u)' - \frac{1}{2.3}(f^3u)'' + \frac{1}{2.3.4}(f^4u)''' - \ldots.,$$

we have $\quad fu = S_1u^3 + S_2u^5 + S_3u^7 + S_4u^9 + \ldots.,$

and thence $f^2u = \quad S_1^2u^6 + 2S_1S_2u^8 + (2S_1S_3 + S_2^2)\,u^{10}\ldots.,$

$f^3u = \quad S_1^3u^9 + 3S_1^2S_2u^{11}\ldots.,$

$f^4u = \quad S_1^4u^{12};$

consequently,

$$\begin{aligned} y = {} & u, \\ & + u^3(-S_1), \\ & + u^5(-S_2 + 3S_1^2), \\ & + u^7(-S_3 + 8S_1S_2 - 12S_1^3), \\ & + u^9(-S_4 + 10S_1S_3 + 5S_2^2 - 55S_1^2S_2 + 55S_1^4), \end{aligned}$$

. . . .

. . . .

and writing herein

$$
\begin{aligned}
u &= \rho\,\{x + R_1x^3 + R_2x^5 + R_3x^7 + R_4x^9 \ldots\ldots\},\\
u^3 &= \rho^3\{x^3 + 3R_1x^5 + (3R_2 + 3R_1^2)\,x^7 + (3R_3 + 6R_1R_2 + R_1^3)\,x^9 \ldots\ldots\},\\
u^5 &= \rho^5\{x^5 + 5R_1x^7 + (5R_2 + 10R_1^2)\,x^9 \ldots\ldots\},\\
u^7 &= \rho^7\{x^7 + 7R_1x^9 \ldots\ldots\},\\
u^9 &= \rho^9\{x^9 + \ldots\ldots\},
\end{aligned}
$$

we have the required series

$$y = \rho x\,\{1 + \Pi_1x^2 + \Pi_2x^4 + \Pi_3x^6 + \ldots\ldots\},$$

where the values of the coefficients are

$$
\begin{aligned}
\Pi_1 &= R_1 + (-S_1)\,\rho^2,\\
\Pi_2 &= R_2 + (-S_1)\,3R_1\rho^2 + (-S_2 + 3S_1^2)\,\rho^4,\\
\Pi_3 &= R_3 + (-S_1)(3R_2 + 3R_1^2)\,\rho^2 + (-S_2 + 3S_1^2)\,5R_1\rho^4 + (-S_3 + 8S_1S_2 - 12S_1^3)\,\rho^6,\\
\Pi_4 &= R_4 + (-S_1)(3R_3 + 6R_1R_2 + R_1^3)\,\rho^2 + (-S_2 + 3S_1^2)(5R_2 + 10R_1^2)\,\rho^4\\
&\quad + (-S_3 + 8S_1S_2 - 12S_1^3)\,7R_1\rho^6 + (-S_4 + 10S_1S_3 + 5S_2^2 - 55S_1^2S_2 + 55S_1^4)\,\rho^8,
\end{aligned}
$$

. . . .

. . . .

and so on as far as we please.

The Cubic Transformation, $n = 3$. *Art. Nos.* 21 *to* 28.

21. We have here

$$\frac{\rho + x^2}{1 + \rho x^2} = \rho\,(1 + \Pi_1x^2 + \Pi_2x^4 + \ldots\ldots);$$

whence, developing the left-hand side and equating coefficients,

$$\rho\Pi_1 = -\rho^2 + 1,\quad \rho\Pi_2 = \rho^3 - \rho,\quad \rho\Pi_3 = -\rho^4 + \rho^2,\ \ldots\ldots$$

It will be convenient to write

$$
\begin{aligned}
\Theta_1 &= \rho\Pi_1 + \rho^2 - 1, \;= -S_1\rho^3 + \rho^2 + R_1\rho - 1,\\
\Theta_2 &= \Pi_2 - \rho^2 + 1, \;= (-S_2 + 3S_1^2)\,\rho^4 - (3R_1S_1 + 1)\,\rho^2 + R_2 + 1,\\
\Theta_3 &= \Pi_3 + \rho^3 - \rho, \;= (-S_3 + 8S_1S_2 - 12S_1^3)\,\rho^6\\
&\qquad + (-5R_1S_2 + 15R_1S_1^2)\,\rho^4\\
&\qquad + \rho^3\\
&\qquad + (-3R_2S_1 - 3R_1^2S_1)\,\rho^2\\
&\qquad - \rho\\
&\qquad + R_3,
\end{aligned}
$$

. . . .

. . . .

where observe the difference of form in the function Θ_1, and in the subsequent functions $\Theta_2, \Theta_3, \ldots$. In these last a factor ρ is thrown out.

22. The two equations $\Theta_1 = 0$ and $\Theta_2 = 0$ serve to determine ρ, β in terms of α; the subsequent equations $\Theta_3 = 0$, $\Theta_4 = 0, \ldots$ will then be, all of them, satisfied identically. This implies that $\Theta_3, \Theta_4, \ldots$ are each of them a linear function of Θ_1, Θ_2. The *à posteriori* verification and determination of the factors is by no means easy; I have effected it only for Θ_3; we have

$$7\Theta_3 = (\rho^3 - 3S_1\rho^2 - 2\rho + 27R_1)\,\Theta_1 + (-S_1\rho^2 - 10\rho + 25R_1)\,\Theta_2,$$

or, at full length,

$$7\left|\begin{array}{r} (-S_3 + 8S_1S_2 - 12S_1^3)\,\rho^6 \\ +(-5R_1S_2 \quad + 15R_1S_1^2)\,\rho^4 \\ +\rho^3 \\ +(-3R_2S_1 \quad - 3R_1^2S_1)\,\rho^2 \\ -\rho \\ +R_3 \end{array}\right|$$

$$\begin{aligned} = {} & (\rho^3 - 3S_1\rho^2 - 2\rho + 27R_1)(-S_1\rho^3 + \rho^2 + R_1\rho - 1) \\ & + (-S_1\rho^2 - 10\rho + 25R_1)((-S_2 + 3S_1^2)\,\rho^4 - (3R_1S_1 + 1)\,\rho^2 + R_2 + 1); \end{aligned}$$

in verifying which we must, of course, take account of the relations between the expressions R and those between the expressions S; we have

$$\alpha = 3R_1 \text{ and thence } 10R_2 = 27R_1^2 - 1,\ \ 14R_3 = 135R_1^3 - 9R_1;$$

similarly,

$$10S_2 = 27S_1^2 - 1,\ \ 14S_3 = 135S_1^3 - 9S_1.$$

Equating the coefficients of ρ^6, we have

$$-7S_3 + 56S_1S_2 - 84S_1^3 = -S_1 + S_1S_2 - 3S_1^3;$$

viz., multiplying by 2, this is

$$-14S_3 + 110S_1S_2 - 162S_1^3 + 2S_1 = 0;$$

or, finally, it is

$$(-135S_1^3 + 9S_1) + (297S_1^3 - 11S_1) - 162S_1^3 + 2S_1 = 0,$$

an identity, as it should be. The identity of the coefficients of $\rho^5, \rho^4, \rho^3, \rho^2, \rho, 1$ may be verified in like manner.

23. Considering α as known, the values of ρ and β are determined by the foregoing equations $\Theta_1 = 0$, $\Theta_2 = 0$; that is,

$$-S_1\rho^3 + \rho^2 + R_1\rho - 1 = 0,$$

$$(-S_2 + 3S_1^2)\,\rho^4 - (3R_1S_1 + 1)\,\rho^2 + R_2 + 1 = 0,$$

(where, of course, the R's and S's are regarded as given functions of α and β respectively).

It is to be observed that the equations are satisfied by $\rho^2 = 1$, $\alpha = \beta$; viz., we have the transformation $y = \frac{x(\pm 1 + x^2)}{1 \pm x^2}$; that is, $y = \pm x$, which is the transformation of the first order, $n = 1$. The two equations represent surfaces of the orders 4 and 6 respectively, and they have thus a complete intersection of the order 24. As part of this, we have, as just shown, each of the two lines $(\rho = 1,\ \alpha = \beta)$ and $(\rho = -1,\ \alpha = \beta)$; but there is a more considerable reduction of order to be accounted for, the proper MM-curve being, as will appear, a unicursal curve of the order $= 6$.

24. Multiplying the second equation by $10\rho^2$, and for $10R_3$, $10S_3$, writing their values $27R_1^2 - 1$ and $27S_1^2 - 1$ respectively, we have

$$(3S_1^2 + 1)\rho^6 - (30R_1S_1 + 10)\rho^4 + (27R_1^2 + 9)\rho^2 = 0;$$

and if herein we substitute for $S_1\rho^3$ its value from the first equation, $= \rho^2 + R_1\rho - 1$, we have

$$3(\rho^2 + R_1\rho - 1)^2 + \rho^6 - 30R_1\rho(\rho^2 + R_1\rho - 1) - 10\rho^4 + (27R_1^2 + 9)\rho^2 = 0;$$

that is, $$\rho^6 - 7\rho^4 - 24R_1\rho^3 + 3\rho^2 + 24R_1\rho + 3 = 0;$$

viz., this is $$(\rho^2 - 1)(\rho^4 - 6\rho^2 - 24R_1\rho - 3) = 0,$$

containing, as it ought to do, the factor $\rho^2 - 1$. Throwing this out, and repeating the first equation, we have

$$\begin{aligned} -S_1\rho^3 + \rho^2 + R_1\rho - 1 &= 0, \\ \rho^4 \qquad - 6\rho^2 - 24R_1\rho - 3 &= 0, \end{aligned}$$

which two equations may be replaced by

$$\begin{aligned} \rho^4 - 24S_1\rho^3 + 18\rho^2 \qquad\qquad - 27 &= 0, \\ \rho^4 \qquad\qquad - 6\rho^2 - 24R_1\rho - 3 &= 0, \end{aligned}$$

which are the $\rho\beta$- and $\rho\alpha$-equations respectively. Recollecting that R_1 and S_1 denote $\frac{1}{3}\alpha$ and $\frac{1}{3}\beta$, they agree with the results obtained in No. 7. The $\alpha\beta$-modular equation is obtained by the elimination of ρ from these two equations, and may be at once written down in the form, Det. $= 0$, where the determinant is of the order 8, but contains S_1 and R_1, that is, β and α, each of them, in the fourth order only: the form is thus the same with that of the $\alpha\beta$-equation obtained in No. 2; but the identification would be a work of some labor.

25. The equations may be written

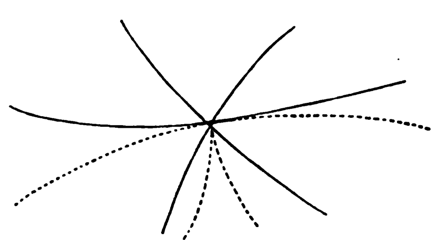
$$24S_1\rho^3 = \rho^4 + 18\rho^2 - 27,$$
$$24R_1\rho = \rho^4 - 6\rho^2 - 3,$$

and, treating R_1, S_1, ρ as coordinates, it hence appears that the MM-curve is (as mentioned above) a unicursal curve of the order 6; in fact, we have R_1, S_1, each of them given as a rational function of ρ; and cutting the curve by an arbitrary plane $AR_1 + BS_1 + C\rho + D = 0$, the substitution of the values of R_1, S_1 in this equation gives for ρ an equation of the order 6.

26. The same conclusion may be obtained from the foregoing system of a cubic and a quartic equation in ρ. Considering R_1, S_1, ρ as coordinates, they represent, each of them, a surface of the order 4, and the complete intersection is of the order 16; but this is made up of a line in the plane infinity counting 10 times, and of the MM-curve, which is thus of the order $16 - 10, = 6$. In fact, introducing, for homogeneity, a fourth co-ordinate θ, the two equations are

$$\begin{array}{rl} -S_1\rho^3 + \rho^2\theta^2 + R_1\rho\theta^3 - \theta^4 = 0, \\ \rho^4 - 6\rho^2\theta^2 - 24R_1\rho\theta^3 - 3\theta^4 = 0, \end{array}$$

and the line $\rho = 0$, $\theta = 0$ is thus a triple line on each of these surfaces; viz., cutting them by an arbitrary plane, we have for the first surface an ordinary triple point, as shown by the continuous lines of the annexed figure, and for the second surface a triple point = cusp + two nodes, as shown by the dotted lines of the figure. There is, moreover, as shown in the figure, a contact of two branches, and the number of intersections is thus = 10.



27. If we assume $\rho\sigma = -3$, that is, $\rho = -\frac{3}{\sigma}$, and substitute this value in the equation for S_1, the two equations become

$$24S_1\sigma = \sigma^4 - 6\sigma^2 - 3,$$
$$24R_1\rho = \rho^4 - 6\rho^2 - 3;$$

viz., β' is the same function of $\sigma\left(= -\frac{3}{\rho}\right)$ which α is of ρ. This accords with

the theorem in Elliptic Functions that a combination of two transformations leads to a multiplication.

28. We have

$$24\left(R_1+\frac{1}{3}\right)\rho=\rho^4-6\rho^2+8\rho-3,\ =(\rho-1)^3(\rho+3);$$

or, what is the same thing,

$$24\left(R_1+\frac{1}{3}\right)=(\rho-1)^2(\rho+\sigma+2);$$

and, in like manner,

$$24\left(R_1-\frac{1}{3}\right)\rho=\rho^4-6\rho^2-8\rho-3,\ =(\rho+1)^3(\rho-3);$$

and, consequently,

$$24\left(R_1-\frac{1}{3}\right)=(\rho+1)^2.(\rho+\sigma-2),$$

with the like equations between S_1, σ, ρ. It will be recollected that

$$R_1=\frac{1}{3}\alpha,\ =\frac{1}{6}\left(u^4+\frac{1}{u^4}\right);$$

hence

$$24\left(R_1\pm\frac{1}{3}\right)=4\left(u^4+\frac{1}{u^4}\pm 2\right),\ =4\left(u^2-\frac{1}{u^2}\right)^2.$$

The formulæ just obtained are useful for obtaining the uv-modular equation from the foregoing equations; or say

$$4\left(v^4+\frac{1}{v^4}\right)\sigma=\sigma^4-6\sigma^2-3,$$

$$4\left(u^4+\frac{1}{u^4}\right)\rho=\rho^4-6\rho^2-3,$$

where $\rho\sigma=-3$, and we have to eliminate ρ and σ; the elimination gives $\frac{v^2}{u^2}-\frac{u^2}{v^2}+2vu-\frac{2}{uv}=0$; that is, $v^4+2v^3u^3-2vu-u^4=0$.

The Quintic Transformation, $n=5$. Art. Nos. 29 to 32.

29. We have here

$$\frac{\rho+A_1x^2+x^4}{1+A_1x^2+\rho x^4}=\rho\left(1+\Pi_1x^2+\Pi_2x^4+\Pi_3x^6+\ldots.\right),$$

and multiplying by $1 + A_1x^2 + \rho x^4$, we obtain an infinite series of equations, the first three of which are

$$A_1 = \rho\Pi_1 + A_1\rho,$$
$$1 = \rho\Pi_2 + A_1\rho\Pi_1,$$
$$0 = \rho\Pi_3 + A_1\rho\Pi_2 + \rho^2\Pi_1,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$

The first of these gives

$$A_1 = \frac{\rho\Pi_1}{-\rho+1}, = \frac{\Theta_1 - \rho^2 + 1}{-\rho+1},$$

and the other two equations then determine the MM-curve. These being satisfied, the remaining equations will be satisfied identically. It is proper to introduce into the equations $\Theta_1, \Theta_2, \Theta_3$ instead of Π_1, Π_2, Π_3. We have first

$$1 = \rho\Pi_2 + \rho\Pi_1\frac{\Theta_1 - \rho^2 + 1}{-\rho+1} + \rho^2;$$

that is,

$$0 = \rho(\Theta_2 + \rho^2 - 1) - \frac{(\Theta_1 - \rho^2 + 1)^2}{\rho - 1} + \rho^2 - 1;$$

viz., this is

$$\rho(\rho-1)(\Theta_2 + \rho^2 - 1) - (\Theta_1 - \rho^2 + 1)^2 + (\rho^2 - 1)(\rho - 1) = 0;$$

or, finally, this is

$$\rho(\rho-1)\Theta_2 - \Theta_1^2 + 2\Theta_1(\rho^2 - 1) = 0.$$

Secondly, we have

$$0 = \Pi_3 + \frac{\rho\Pi_1\Pi_2}{-\rho+1} + \rho\Pi_1 = 0;$$

that is,

$$\Theta_3 - \rho^3 + \rho + \frac{(\Theta_1 - \rho^2 + 1)(\Theta_2 + \rho^2 - 1)}{-\rho+1} + \Theta_1 - \rho^2 + 1 = 0;$$

viz., this is

$$(\Theta_3 + \Theta_1 - \rho^3 - \rho^2 + \rho + 1)(-\rho + 1) + (\Theta_1 - \rho^2 + 1)(\Theta_2 + \rho^2 - 1) = 0;$$

or, finally, it is

$$\Theta_3(-\rho + 1) + \Theta_1\Theta_2 + \Theta_1(\rho^2 - \rho) - \Theta_2(\rho^2 - 1) = 0.$$

30. We have thus the two equations

$$(\rho^2 - \rho)\Theta_2 - \Theta_1^2 + 2\Theta_1(\rho^2 - 1) = 0,$$
$$\Theta_3(-\rho + 1) + \Theta_1\Theta_2 + \Theta_1(\rho^2 - \rho) - \Theta_2(\rho^2 - 1) = 0;$$

and recollecting that Θ_3 is of the form $L\Theta_1 + M\Theta_2$, we see that each of these equations is satisfied if only $\Theta_1 = 0$, $\Theta_2 = 0$ (the formulæ belonging to the cubic transformation). This ought to be the case, for we can, by writing $A_1 = \rho + 1$,

reduce the expression $\dfrac{x(\rho+A_1x^2+x^4)}{1+A_1x^2+\rho x^4}$ to the form $\dfrac{x(\rho+x^2)}{1+\rho x^2}$, which belongs to the cubic transformation (see *ante* No. 17). The equations may be written

$$\rho\Theta_2 = -(2\rho+2)\Theta_1 + \frac{\Theta_1^2}{\rho-1},$$

$$\rho\Theta_3 = (3\rho^2+4\rho+2)\Theta_1 - (3\rho+3)\frac{\Theta_1^2}{\rho-1} + \frac{\Theta_1^3}{(\rho-1)^2}.$$

31. The investigation may be presented in a slightly different form by introducing the functions Θ at an earlier stage; viz., writing $\rho\Pi_1 = \Theta_1 - \rho^2 + 1$, $\rho\Pi_2 = \rho\Theta_2 + \rho^3 - \rho, \ldots\ldots$, we have

$$\frac{\rho+A_1x^2+x^4}{1+A_1x^2+\rho x^4} = \rho + (\Theta_1-\rho^2+1)x^2 + (\rho\Theta_2+\rho^3-\rho)x^4 + \ldots\ldots$$

$$= \frac{\rho+x^2}{1+\rho x^2} + \Theta_1x^2 + \rho\Theta_2x^4 + \rho\Theta_3x^6 + \ldots\ldots$$

Transposing, reducing, and dividing by x^2, we have

$$\frac{(1-x^2)[\rho^2-1+A_1(-\rho+1)]}{(1+\rho x^2)(1+A_1x^2+\rho x^4)} = \Theta_1 + \rho\Theta_2x^2 + \rho\Theta_3x^4 + \ldots\ldots,$$

whence clearly $\rho^2-1+A_1(-\rho+1) = \Theta_1$, giving for A_1 the before-mentioned value; and we then have

$$1+A_1x^2+\rho x^4 = 1+(\rho+1)x^2+\rho x^4 - \frac{\Theta_1x^2}{\rho-1}, = (1+x^2)(1+\rho x^2) - \frac{\Theta_1x^2}{\rho-1}.$$

The equation thus becomes

$$\frac{(1-x^2)\Theta_1}{(1+x^2)(1+\rho x^2)^2\left(1-\dfrac{\Theta_1x^2}{\rho-1.1+x^2.1+\rho x^2}\right)} = \Theta_1 + \rho\Theta_2x^2 + \rho\Theta_3x^4 + \ldots\ldots,$$

and expanding the left-hand side, first in the form

$$\frac{(1-x^2)\Theta_1}{(1+x^2)(1+\rho x^2)^2} + \frac{(1-x^2)x^2\Theta_1^2}{(\rho-1)(1+x^2)^2(1+\rho x^2)^3} + \frac{(1-x^2)x^4\Theta_1^3}{(\rho-1)(1+x^2)^3(1+\rho x^2)^4} + \ldots\ldots,$$

and then each of these terms separately in powers of x^2, and comparing with $\Theta_1 + \rho\Theta_2x^2 + \rho\Theta_3x^4 + \ldots\ldots$, we have the two equations in the last-mentioned form, and an infinite series of other equations, which will be satisfied identically.

32. The successive coefficients might be called $\Phi_2, \Phi_3, \ldots\ldots$; say

$$\Phi_2 = (\rho^2-\rho)\Theta_2 - \Theta_1^2 + 2(\rho^2-1)\Theta_1,$$

$$\Phi_3 = (-\rho+1)\Theta_3 + \Theta_1\Theta_2 + (\rho^2-\rho)\Theta_1 - (\rho^2-1)\Theta_2,$$

and similarly for $\Phi_4, \ldots\ldots$, and it would then be proper to show *à posteriori* that

each of the equations $\Phi_4=0$, $\Phi_5=0$, ... is satisfied identically in virtue of the two equations $\Phi_2=0$, $\Phi_3=0$, or, what is the same thing, that the functions Φ_4, Φ_5, ... are each of them a linear function (with coefficients which are functions of ρ) of the two functions Φ_2 and Φ_3. I do not attempt to do this, nor even to discuss the MM-curve by means of the equations $\Phi_2=0$, $\Phi_3=0$; but I will obtain equivalent results, and complete the solution by means of the Jacobi-Brioschi equations, in effect reproducing the investigation contained in the third appendix of the Funzioni Ellittiche.

The General Transformation, $n=2s+1$. Art. No. 33.

33. The equation here is

$$\frac{\rho+A_{s-1}x^2+\ldots.}{1+A_1x^2+\ldots.}=\rho(1+\Pi_1x^2+\ldots.).$$

The general theory is sufficiently illustrated by the preceding particular cases, and I wish at present only to notice the equation obtained by comparing the coefficients of x^2; viz., this is $A_{s-1}-\rho A_1=\rho\Pi_1$, or, substituting for Π_1 its value,

$$A_{s-1}-\rho A_1=\frac{1}{3}(\alpha\rho-\beta\rho^3).$$

The Jacobi-Brioschi Equations. Art. Nos. 34 to 42.

34. These were obtained for the differential equation

$$\frac{dx}{\sqrt{a'x^4+b'x^3+c'x^2+d'x+e'}}=\frac{dy}{\sqrt{ay^4+by^3+cy^2+dy+e}};$$

viz., if this be satisfied by $y=U\div V$, where U, V are rational and integral functions of x of the degrees n and $n-1$ respectively, then, writing for shortness $\phi=a'x^4+b'x^3+c'x^2+d'x+e'$, and using accents to denote differentiation in regard to x, the numerator and denominator U, V satisfy the equations

$$\begin{aligned}
(VV''-V'^2)\phi+\frac{1}{2}VV'.\phi'\quad &+\ aU^2+\ \frac{1}{2}bUV+\ pV^2=0,\\
-(VU''+V''U-2V'U')\phi-\frac{1}{2}(VU'+V'U)\phi'&+\frac{1}{2}bU^2+(c-2p)UV+\frac{1}{2}dV^2=0,\\
(UU''-U'^2)\phi+\frac{1}{2}UU'.\phi'\quad &+\ pU^2+\ \frac{1}{2}dUV+\ eV^2=0,
\end{aligned}$$

where p is a function $=\mathrm{a}x^2+\mathrm{b}x+\mathrm{c}$, with coefficients a, b, c, the values of

which have to be determined. By way of verification, observe that, multiplying by U^2, UV, V^2, and adding, we obtain

$$-(VU'-V'U)^2\phi + aU^4 + bU^3V + cU^2V^2 + dUV^3 + eV^4 = 0;$$

that is,

$$-\frac{1}{V^2}(VU'-V'U)^2(a'x^4+b'x^3+c'x^2+d'x+e') + ay^4 + by^3 + cy^2 + dy + e = 0,$$

the result obtained by substituting for y its value, $= U \div V$, in the differential equation.

35. Considering the foregoing special form

$$\frac{dx}{\sqrt{1-2ax^2+x^4}} = \frac{dy}{\rho\sqrt{1-2\beta y^2+y^4}},$$

so that a, b, c, d, e have the values ρ^2, 0, $-2\beta\rho^2$, 0, ρ^2 and ϕ is $= 1-2ax^2+x^4$, the equations are

$$\begin{aligned}
(VV''-V'^2)\phi + {}& \tfrac{1}{2}VV'.\phi' + \rho^2U^2 + pV^2 = 0,\\
-(VU''+V''U-2V'U')\phi + {}& \tfrac{1}{2}(VU'+V'U)\phi' - (2\beta\rho^2+2p)UV = 0,\\
(UU''-U'^2)\phi + {}& \tfrac{1}{2}UU'.\phi' + pU^2 + \rho^2V^2 = 0,
\end{aligned}$$

where, writing as before, $n = 2s+1$, and assuming that the last coefficient, $A_{\frac{1}{2}(n-1)}$ or A_s, is $= \rho$, we have

$$\begin{aligned}
U &= x(\rho + A_{s-1}x^2 + A_{s-2}x^4 \ldots + A_1x^{2s-2} + x^{2s}),\\
V &= 1 + A_1x^2 + A_2x^4 \ldots + A_{s-1}x^{2s-2} + \rho x^{2s},
\end{aligned}$$

and where, as is easily shown, p has the value $= -\{2A_1 + (2s+1)x^2\}$. In comparing with Brioschi, it will be recollected that 2α, 2β are written in place of his α, β.

36. The equations contain n, and they are not satisfied by values of U, V belonging to any inferior value of n; U, V may each of them be multiplied by any common constant factor at pleasure, but not by a common variable factor P; viz., it is assumed that the fraction $U \div V$ is in its least terms, and consequently that (save as to a constant factor) U, V are determinate functions. It is easy to verify that the equations (being verified by U, V) are not verified by PU, PV, but it is interesting to show *à priori* why this is so. The equations are obtained as follows: Consider the differential equation in the form $\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$,

and suppose that an integral equation is given in the form $F=0$ (F a rational and integral function of x, y); we thence deduce a relation $Ldx+Mdy=0$ between the differentials, and this must agree with the given differential equation; that is, we have $L\sqrt{X}+M\sqrt{Y}=0$, or, rationalizing, $L^2X-M^2Y=0$; viz., this last equation must agree with the equation $F=0$, or, what is the same thing, L^2X-M^2Y must contain F as a factor; say we have

$$L^2X-M^2Y=F.G,$$

where G is a function of x, y. In the particular case where the integral is of the form $y=U\div V$, we have $F=Vy-U$,

and we have therefore $\quad L^2X-M^2Y=G(Vy-U)$;

and it is by means of this identity that the equations are obtained. But suppose that there is a common factor P, or that we have $y=PU\div PV$; then, if we write $F=PVy-PVU$, $=P(Vy-U)$, there is no necessity that L^2X-M^2Y should contain as a factor this expression of F, and it will not in fact contain it; all that is necessary is that L^2X-M^2Y shall contain the factor $Vy-U$; and thus the equations obtained for U, V do not apply to PU, PV. We might, of course, introduce an arbitrary *constant* factor Θ; contrast herewith the solution by means of the Jacobi partial differential equation, *post* No. 43, where Θ is not arbitrary, but has a determinate value.

37. In virtue of the assumed forms of U, V, the first and third equation give each of them the same relations between the coefficients A; and only one of these equations, say the first, need be attended to. It will be observed that this equation does not contain β; it consequently serves to determine the coefficients A in terms of ρ, α, and to establish a relation between ρ, α; that is, the multiplier equation. We can from this, as will be explained, deduce the equation between ρ, β; the theory thus depends entirely upon the first equation; say this is

$$(VV''-V'^2)(1-2\alpha x^2+x^4)+VV'(-2\alpha x+2x^3)+\rho^2U'^2-\{2A_1+(2s+1)x^2\}V^2=0.$$

38. We have $V=1+A_1x^2+A_2x^4+\ldots.$, but the equation contains the quadric functions $VV''-V'^2$, VV', and V^2; it is convenient to write

$$VV''-V'^2=K_1+K_2x^2+K_3x^4+\ldots.,$$
$$V^2=L_0+L_1x^2+L_2x^4+\ldots.;$$

whence of course $\quad VV'=\quad 2L_1x+4L_2x^3+\ldots.,$

and we have

$K_1=$	$K_2=$	$K_3=$	$K_4=$	$K_5=$	$K_6=$	$K_7=$	$K_8=$	....
$2A_1$	$12A_2$	$30A_3$	$56A_4$	$90A_5$	$132A_6$	$182A_7$	$240A_8$	
	$-2A_1^2$	$-2A_1A_2$	$+8A_1A_3$	$+26A_1A_4$	$+52A_1A_5$	$+86A_1A_6$	$+128A_1A_7$	
			$-\ 4A_2^2$	$-\ 6A_2A_3$	$+\ 4A_2A_4$	$+22A_2A_5$	$+\ 48A_2A_6$	,
					$-\ 6A_3^2$	$-10A_3A_4$	$0A_3A_5$	
							$-\ 8A_4^2$	

$L_0=$	$L_1=$	$L_2=$	$L_3=$	$L_4=$	....
1	$2A_1$	$2A_2$	$2A_3$	$2A_4$	
		$+\ A_2^2$	$+2A_1A_2$	$+2A_1A_3$	.
				$+\ A_2^2$	

The coefficients of U^2 are at once obtained; say we have $U^2=\Lambda_0x^2+\Lambda_1x^4+\Lambda_2x^6\ldots,$

$\Lambda_0=$	$\Lambda_1=$	$\Lambda_2=$	$\Lambda_3=$	$\Lambda_4=$	....
ρ^2	$2\rho A_{s-1}$	$2\rho A_{s-2}$	$2\rho A_{s-3}$	$2\rho A_{s-4}$	
		$+A_{s-1}^2$	$+2A_{s-1}A_{s-2}$	$+2A_{s-1}A_{s-3}$	.
				$+\ A_{s-2}^2$	

Substituting in the equation and equating to zero the coefficients of the several powers of x^2, we find

$$
\begin{aligned}
&K_1 && -2A_1L_0 && && && =0,\\
&K_2 && -2A_1L_1+(-2s-1)\,L_0 && -2\alpha\,(K_1+\ L_1) && +\rho^2\Lambda_0 && =0,\\
&K_3+K_1 && -2A_1L_2+(-2s+1)\,L_1 && -2\alpha\,(K_2+2L_2) && +\rho^2\Lambda_1 && =0,\\
&K_4+K_2 && -2A_1L_3+(-2s+3)\,L_2 && -2\alpha\,(K_3+3L_3) && +\rho^2\Lambda_2 && =0,\\
&K_5+K_3 && -2A_1L_4+(-2s+5)\,L_3 && -2\alpha\,(K_4+4L_4) && +\rho^2\Lambda_3 && =0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

The number of equations is $=2\,(s+1)$, for the equation contains terms in $x^0, x^2, x^4, \ldots.\ x^{4s+2}$; but the first equation, and also the last and last but one equations, are in fact identities; there remain thus $2(s+1)-3,\ =2s-1$ equations; but these are equivalent to s independent equations, serving to determine the $(s-1)$ coefficients $A_1, A_2, \ldots.\ A_{s-1}$, and to determine the relation between ρ and α. In writing down the equations for a determinate value of s, the coefficients

A_0, A_s must be taken to be $=0$ and ρ respectively; and coefficients with a negative suffix or a suffix greater than s, must be taken to be each $=0$.

39. Thus, $(n=3)\, s=1$, we have the $2(s+1),\ =4$ equations:

$$\begin{array}{llllll}
2\rho & -2\rho.1 & & & & =0, \\
-2\rho^2 & -2\rho.2\rho & +(-3)\,1 & -2\alpha\,(\ \ 2\rho+2\rho) & +\rho^2.\rho^2 & =0, \\
0+2\rho & -2\rho.\rho^2 & +(-1)\,2\rho & -2\alpha\,(-2\rho^2+2\rho^2) & +\rho^2.2\rho & =0, \\
0-2\rho^2 & -2\rho.0 & +(+1)\,\rho^2 & -2\alpha\,(\ \ 0+3.0) & +\rho^2.1 & =0,
\end{array}$$

where the first, third and fourth equations are each of them an identity; the second equation is $-2\rho^2-4\rho^2-3-8\alpha\rho+\rho^4=0$; viz., in accordance with what precedes, writing $\alpha=3R_1$, this is the foregoing equation

$$\rho^4-6\rho^2-24R_1\rho-3=0.$$

To complete the solution, we use the theorem in elliptic functions referred to *ante* (No. 8); viz., writing $\rho\sigma=-3$, then we have β, the same function of σ that α is of ρ; whence

$$\sigma^4-6\sigma^2-24S_1\sigma-3=0,$$

and we thus have two equations giving the MM-curve.

40. In the case $(n=5)\, s=2$, we have the $2(s+1),\ =6$ equations:

$$\begin{array}{llllll}
2A_1 & -2A_1.1 & & & & =0, \\
12\rho-2A_1^2 & -2A_1(2A_1) & -5.1 & -2\alpha.2A_1 & +\rho^2.\rho^2 & =0, \\
-2\rho A_1+2A_1 & -2A_1(2\rho+A_1^2) & -3.2A_1 & -2\alpha\{12\rho-2A_1^2+2(2\rho+A_1^2)\} & +\rho^2.2\rho A_1 & =0, \\
-4\rho^3+12\rho & -2A_1^2-2A_1.2A_1\rho & -1(2\rho+A_1^2) & -2\alpha\{-2\rho A_1+3.2A_1\rho\} & +\rho^2(2\rho+A_1^2) & =0, \\
0 & -2\rho A_1-2A_1.\rho^2 & +1.2A_1\rho & -2\alpha\{-4\rho^2\ \ +4.\rho^2\} & +\rho^2.2A_1 & =0, \\
0 & -4\rho^2\ \ -2A_1.0 & +3.\rho^2 & -2\alpha\{\ \ 0\ \ +5.0\} & +\rho^2.1 & =0,
\end{array}$$

where the first, fifth and sixth equations are each of them an identity. The remaining equations are

$$(\rho^2-2\rho+5)(\rho^2+2\rho-1)-6A_1^2-8A_1\alpha=0,$$
$$2\rho^3A_1-6\rho A_1-32\rho\alpha-2A_1^3-4A_1=0,$$
$$2\rho(\rho^2-2\rho+5)+10\rho-4A_1^2\rho-8\alpha A_1\rho+3A_1^2=0.$$

41. Writing the first and third of these in the forms

$$\begin{array}{llll}
-6A_1^2 & -8A_1\alpha & +(\rho^2-2\rho+5)(\rho^2+2\rho-1) & =0, \\
A_1^2(\rho^2-4\rho+3) & -8A_1\alpha\rho & +(\rho^2-2\rho+5)\,2\rho & =0,
\end{array}$$

these determine A_1^2, $8A_1\alpha$ in terms of ρ; viz., we find

$$A_1^2=(\rho^2-2\rho+5)\,\rho,$$
$$8A_1\alpha=(\rho^2-2\rho+5)(\rho^2-4\rho-1);$$

and then writing the second equation in the form

$$(\rho^3 - 3\rho - 2)\,A_1^2 - 16\rho\alpha A_1 - A_1^4 = 0,$$

and substituting these values of A_1^2 and $8A_1\alpha$, and omitting the factor $\rho^2 - 2\rho - 5$, we have the identity

$$\rho\,(\rho^3 - 3\rho - 2) - 2\rho\,(\rho^2 - 4\rho - 1) - \rho^2(\rho^2 - 2\rho + 5) = 0;$$

viz., the second equation is then also satisfied.

Forming the square of $8A_1\alpha$, and for A_1^2 substituting its value, then omitting a factor $\rho^2 - 2\rho + 5$, we find

$$\begin{aligned} 64\rho\alpha^2 &= (\rho^2 - 2\rho + 5)(\rho^2 - 4\rho - 1)^2, \\ &= \rho^6 - 10\rho^5 + 35\rho^4 - 60\rho^3 + 55\rho^2 + 38\rho + 5; \end{aligned}$$

or, as this may also be written,

$$64\rho\,(\alpha^2 - 1) = (\rho - 1)^5(\rho - 5),$$

and we then have also, as before,

$$64\sigma\,(\beta^2 - 1) = (\sigma - 1)^5(\sigma - 5),$$

which two equations determine the MM-curve.

The coefficient A_1 is given by the foregoing equation for $8A_1\alpha$, say the value is

$$A_1 = \frac{1}{8\alpha}\,(\rho^2 - 2\rho + 5)(\rho^2 - 4\rho - 1).$$

The value $A = \dfrac{\rho\Pi_1}{-\rho+1}$, obtained in No. 28, substituting for Π_1 its value, is

$$A_1 = \frac{\frac{1}{3}\,(\beta\rho^3 - \alpha\rho)}{\rho - 1},$$

and these two values are in fact equivalent in virtue of the value of β obtained in No. 9.

42. I consider the case $n = 7$, in order to show the form of the equations which have to be solved; these equations are

$$\begin{aligned}
&2A_1 - 2A_1.1 &&= 0,\\
&12A_2 - 2A_1^2 - 2A_1.\,2A_1 - 7.1 - 2\alpha\,(2A_1 + 1.2A_1) + \rho^3.\rho^3 &&= 0,\\
&30\rho - 2A_1A_2 + 2A_1 - 2A_1\,(2A_2 + A_1^2) - 5.2A_1\\
&\qquad - 2\alpha\,(12A_2 - 2A_1^2 + 2\,(2A_2 + A_1^2)) + \rho^3.\,2\rho A_2 &&= 0,\\
&8A_1\rho - 4A_2^2 + 12A_2 - 2A_1^2 - 2A_1\,(2\rho + 2A_1A_2) - 3\,(2A_2 + A_1^2)\\
&\qquad - 2\alpha\,(30\rho - 2A_1A_2 + 3\,(2\rho + 2A_1A_2)) + \rho^3\,(2\rho A_1 + A_2^2) &&= 0,\\
&-6A_2\rho + 30\rho - 2A_1A_2 - 2A_1\,(2A_1\rho + A_2^2) - 1\,(2\rho + 2A_1A_2)\\
&\qquad - 2\alpha\,(8A_1\rho - 4A_2^2 + 4\,(2A_1\rho + A_2^2)) + \rho^3\,(2\rho + 2A_1A_2) &&= 0,\\
&-6\rho^2 \;\; + 8A_1\rho - 4A_2^2 - 2A_1\,(2A_2\rho) + 1\,(2A_1\rho + A_2^2)\\
&\qquad - 2\alpha\,(-6A_2\rho + 5\,(2A_2\rho)) + \rho^3\,(2A_2 + A_1^2) &&= 0,\\
&0 \quad - 6A_2\rho - 2A_1.\rho^2 + 3\,(2A_2\rho) - 2\alpha\,(-6\rho^2 + 6.\rho^2) + \rho^3.\,2A_1 &&= 0,\\
&0 \quad - 6\rho^2 + 5.\rho^2 - 2\alpha\,(0 + 7.0) + \rho^3.1 &&= 0;
\end{aligned}$$

viz., the first, seventh and eighth equations are satisfied identically, and there remain five equations connecting ρ, α, A_1, A_2.

These equations should lead to the before-mentioned $\alpha\beta$-modular equation

$$\rho^8 - 28\rho^6 - 112\alpha\rho^5 - 210\rho^4 - 224\alpha\rho^3 + (-1484 + 1344\alpha^2)\rho^2 \\ + (-560\alpha + 512\alpha^3)\rho + 7 = 0,$$

and to expressions for A_1, A_2 as rational functions of α, ρ, and should be, all five of them, satisfied by these results; but I do not see how the results are to be worked out; there is, so far as appears, no clue to the discovery of the rational functions of α, ρ.

The Jacobi Partial Differential Equation. Nos. 43 to 48.

43. Writing, as above, 2α in place of Jacobi's α, this is

$$(1 - 2\alpha x^2 + x^4)\frac{d^2z}{dx^2} + (n-1)(2\alpha x - 2x^3)\frac{dz}{dx} + n(n-1)x^2z - 4n(\alpha^2 - 1)\frac{dz}{dx} = 0,$$

satisfied by the numerator and denominator U, V, each of them taken with the same proper value of the coefficient A_0, or, what is the same thing, by the values

$$U = \Theta x(A_s + A_{s-1}x^2 + A_{s-2}x^4 \ldots + A_1x^{2s-2} + x^{2s}),$$
$$V = \Theta(1 + A_1x^2 + A_2x^4 \ldots + A_{s-1}x^{2s-2} + A_sx^{2s}),$$

where now $A_s = \rho$ as before: Θ has its proper value; viz. (disregarding an arbitrary merely numerical factor which might of course be introduced), the value is

$$\Theta = \sqrt{\frac{1}{M}\frac{\lambda'}{k'}},\ = \sqrt{\frac{u^2\rho}{v^2}\frac{\sqrt{1-v^8}}{\sqrt{1-u^8}}},\ = \sqrt{\rho}\,\frac{\sqrt[4]{v^4 - v^{-4}}}{\sqrt[4]{u^4 - u^{-4}}},$$

or, what is the same thing,

$$\Theta = \sqrt{\rho}\sqrt[8]{\frac{\beta^2 - 1}{\alpha^2 - 1}}.$$

If for z we write $\Theta\zeta$, then the equation becomes

$$(1 - 2\alpha x^2 + x^4)\frac{d^2\zeta}{dx^2} + (n-1)(2\alpha x - 2x^3)\frac{d\zeta}{dx} \\ + n(n-1)x^2\zeta - 4n(\alpha^2 - 1)\left(\frac{d\zeta}{d\alpha} + \frac{1}{\Theta}\frac{d\Theta}{d\alpha}\zeta\right) = 0,$$

satisfied by the foregoing values without the factor Θ, or attending only to the denominator, say by the value

$$V = 1 + A_1x^2 + A_2x^4 \ldots + A_{s-1}x^{2s-2} + \rho x^{2s}.$$

44. To calculate the value of $\frac{1}{\Theta}\frac{d\Theta}{da}$, we have

$$\frac{1}{\Theta}\frac{d\Theta}{da}=\frac{\frac{1}{2}}{\rho}\frac{d\rho}{da}+\frac{\frac{1}{4}\beta}{\beta^2-1}\frac{d\beta}{da}-\frac{\frac{1}{4}\alpha}{\alpha^2-1};$$

but it has been seen (No. 10) that we have

$$\frac{d\beta}{da}=\frac{\rho^2}{n}\frac{\beta^2-1}{\alpha^2-1},$$

and the formula thus becomes

$$\frac{1}{\Theta}\frac{d\Theta}{da}=\frac{\frac{1}{2}}{\rho}\frac{d\rho}{da}+\frac{\frac{1}{4n}(\beta\rho^2-n\alpha)}{\alpha^2-1}.$$

We have, as the first of the equations obtained by substituting in the Partial differential equation,

$$2A_1-4n(\alpha^2-1)\frac{1}{\Theta}\frac{d\Theta}{da}=0,$$

and we have hence the value of the first coefficient,

$$A_1=n(\alpha^2-1)\frac{1}{\rho}\frac{d\rho}{da}+\frac{1}{2}(\beta\rho^2-n\alpha);$$

or we may, by means of this result, get rid of the term $\frac{1}{\Theta}\frac{d\Theta}{da}$ from the Partial differential equation; viz., the equation may be written

$$(1-2\alpha x^2+x^4)\frac{d^2\zeta}{dx^2}+(n-1)(2\alpha x-2x^3)\frac{d\zeta}{dx}+\{n(n-1)x^2-2A_1\}\zeta-4n(\alpha^2-1)\frac{d\zeta}{da}=0.$$

Before going further, I remark that the last of the equations obtained by the substitution gives the coefficient A_{s-1}; but this is also given in terms of A_1 by the formula No. 33, $A_{s-1}-\rho A_1=\frac{1}{3}(\alpha\rho-\beta\rho^3)$; combining the two formulæ, we have

$$\begin{aligned}A_1&=\frac{1}{\rho}n(\alpha^2-1)\frac{d\rho}{da}-\frac{1}{2}n\alpha+\frac{1}{2}\beta\rho^2,\\ A_{s-1}&=n(\alpha^2-1)\frac{d\rho}{da}+\left(-\frac{1}{2}n+\frac{1}{3}\right)\alpha\rho+\frac{1}{6}\beta\rho^3.\end{aligned}$$

45. In the case $n=3$, $A_{s-1}=A_0=1$, $A_1=\rho$, and the two equations become

$$3(\alpha^2-1)\frac{d\rho}{da}-\frac{3}{2}\alpha\rho-\rho^2+\frac{1}{2}\beta\rho^3=0,$$

$$3(\alpha^2-1)\frac{d\rho}{da}-1-\frac{7}{6}\alpha\rho+\frac{1}{6}\beta\rho^3=0,$$

each of which is easily verified.

I remark also that in the same case, $(n=3)$, we have

$$\rho^4\frac{\beta^2-1}{\alpha^2-1}=\left(\frac{\rho^2-9}{\rho^2-1}\right)^2,\text{ and thence }\Theta,\ =\surd\rho\sqrt[4]{\frac{\beta^2-1}{\alpha^2-1}}=\sqrt[4]{\frac{\rho^2-9}{\rho^2-1}}\,;$$

hence

$$\frac{1}{\Theta}\frac{d\Theta}{d\rho}=\frac{4\rho}{(\rho^2-1)(\rho^2-9)}\,;$$

and writing the equation $A_1-2n(\alpha^2-1)\dfrac{1}{\Theta}\dfrac{d\Theta}{d\alpha}=0$ in the form

$$\rho-2n(\alpha^2-1)\frac{1}{\Theta}\frac{d\Theta}{d\rho}\frac{d\rho}{d\alpha}=0,$$

we can verify this equation.

46. In the case $n=5$, we have for A_1 two equations, each ultimately giving the foregoing value

$$A_1=\frac{1}{8\alpha}(\rho^2-2\rho+5)(\rho^2-4\rho-1).$$

Moreover, the equation $\Theta=\surd\rho\sqrt[4]{\dfrac{\beta^2-1}{\alpha^2-1}}$ gives, without difficulty, $\Theta=\dfrac{1}{\surd\rho}\dfrac{\rho-5}{\rho-1}$.

47. In the case $n=7$, the formulæ give the two coefficients A_1, A_2; viz., we have

$$A_1=\frac{1}{\rho}\,7(\alpha^2-1)\frac{d\rho}{d\alpha}-\frac{7}{2}\alpha+\frac{1}{2}\beta\rho^2,$$

$$A_2=\quad 7(\alpha^2-1)\frac{d\rho}{d\alpha}-\frac{19}{6}\alpha\rho+\frac{1}{6}\beta\rho^3,$$

where the value of $\dfrac{d\rho}{d\alpha}$ must of course be obtained from the before-mentioned $\rho\alpha$-equation (given in No. 7). I have not considered these results nor endeavored to compare them with the results for this case obtained in the Transformation Memoir, and the addition thereto.

48. Substituting the value $1+A_1x^2+A_2x^4\ldots\ldots+A_{s-1}x^{2s-2}+\rho x^{2s}$ in the last-mentioned form of the Partial differential equation, we obtain

$$\begin{aligned}
2A_1 &= 2A_1,\\
12A_2 &= -\ 4(n-2)\alpha A_1+2A_1^2\ -n(n-1)\qquad +4n(\alpha^2-1)\frac{dA_1}{d\alpha},\\
30A_3 &= -\ 8(n-4)\alpha A_2+2A_1A_2-(n-2)(n-3)A_1+4n(\alpha^2-1)\frac{dA_2}{d\alpha},\\
56A_4 &= -12(n-6)\alpha A_3+2A_1A_3-(n-4)(n-5)A_2+4n(\alpha^2-1)\frac{dA_3}{d\alpha},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The number of equations is of course finite and $= s + 2$, but the last equation is an identity. To obtain the last but one equation, it is convenient to write down the general equation; viz., this is

$$(2r+1)(2r+2)A_{r+1} = -4r(n-2r)\alpha A_r + 2A_1A_r \\ -(n-2r+1)(n-2r+2)A_{r-1} + 4n(\alpha^2-1)\frac{dA_r}{d\alpha};$$

and then writing herein $r = s$, we have

$$0 = -4s(n-2s)\alpha\rho + 2A_1\rho \\ -(n-2s+1)(n-2s+2)A_{s-1} + 4n(\alpha^2-1)\frac{d\rho}{d\alpha};$$

viz., for n substituting its value $2s+1$, the equation is

$$0 = -2(n-1)\alpha\rho + 2A_1\rho - 6A_{s-1} + 4n(\alpha^2-1)\frac{d\rho}{d\alpha}.$$

Recapitulation of Formulæ for the Cases $n = 3$ *and* $n = 5$. *Art. Nos.* 49 *and* 50.

49. In conclusion, it will be convenient to collect the formulæ as follows:

$$n = 3, \qquad y = \frac{x(\rho + x^2)}{1 + \rho x^2}, \quad \Theta = \sqrt{\frac{\rho^2-9}{\rho^2-1}},$$

$$8\alpha\rho = \rho^4 - 6\rho^2 - 3,$$

$$8(\alpha+1)\rho = (\rho-1)^3(\rho+3);\ 8(\alpha-1)\rho = (\rho+1)^3(\rho-3),$$

$$\sigma = -\frac{3}{\rho}, \quad 8\beta\sigma = \sigma^4 - 6\sigma^2 - 3,$$

$$8(\beta+1)\sigma = (\sigma-1)^3(\sigma+3);\ 8(\beta-1)\sigma = (\sigma-1)^3(\sigma+3).$$

$\alpha\beta$-equation, see No. 2.

50. $n = 5$,

$$y = \frac{x(\rho + A_1x^2 + x^4)}{1 + A_1x^2 + \rho x^4}, \quad \Theta = \frac{1}{\sqrt{\rho}}\frac{\rho-5}{\rho-1},$$

$$A_1 = \frac{1}{8\alpha}(\rho^2 - 2\rho + 5)(\rho^2 - 4\rho - 1),$$

$$64\alpha^2\rho = (\rho^2-4\rho-1)^2(\rho^2-2\rho+5);$$

or say

$$8\alpha\sqrt{\rho} = (\rho^2-4\rho-1)\sqrt{\rho^2-2\rho+5},$$

$$64(\alpha^2-1)\rho = (\rho-1)^5(\rho-5),$$

$$\sigma = \frac{5}{\rho}, \quad 64\beta^2\sigma = (\sigma^2-4\sigma-1)^2(\sigma^2-2\sigma+5),$$

$$-8\beta\sqrt{\sigma} = (\sigma^2-4\sigma-1)\sqrt{\sigma^2-2\rho+5},$$

$$64(\beta^2-1)\sigma = (\sigma-1)^5(\sigma-5),$$

$$\frac{\beta}{\alpha} = -\frac{\rho^2+20\rho-5}{\rho^2(\rho^2-4\rho-1)}.$$

$\alpha\beta$-equation, see No. 3.

The $\rho\alpha$-equations for the cases in question, $n = 3$ and $n = 5$, are the so-called Jacobian equations of the fourth and sixth degrees, studied by Brioschi (in the third appendix above referred to) and by others: the foregoing $\alpha\beta$-equations have not (so far as I am aware) been previously obtained; as rationally connected with the $\rho\alpha$-equations, they must belong to the same class of equations.

CAMBRIDGE (England), 18th Dec., 1886.

Forms, Necessary and Sufficient, of the Roots of Pure Uni-Serial Abelian Equations.

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OBJECT OF THE PAPER.

§ 1. An Abelian equation, that is, an irreducible equation in which one root is a rational function of another and of known quantities, may be called *uni-serial* when the roots form a single circulating series. If the equation, say $f(x) = 0$, be of the n^{th} degree, its roots, in the ordinary Abelian notation, are

$$x_1,\ \theta x_1,\ \theta^2 x_1,\ \ldots\ ,\ \theta^{n-1} x_1. \tag{1}$$

§ 2. When the coefficients of θ are rational, in other words, when one root of the equation $f(x) = 0$ is a rational function of another, the equation is a *pure* Abelian. For instance, the irreducible cubic equation

$$x^3 + px + q = 0,$$

in which the coefficients p and q are such that $\sqrt{(-4p^3 - 27q^2)}$ is rational, is a pure Abelian, because, as is well known, one root of the cubic is a rational function of either of the others.

§ 3. The object of the following paper is to investigate the necessary and sufficient forms of the roots of pure uni-serial Abelian equations. First, a Criterion of pure uni-serial Abelianism is established (§ 12–§ 15). A deduction is then given of the necessary and sufficient forms of the roots of pure uni-serial Abelian equations of all prime degrees (§ 16–§ 26). Then the necessary and sufficient forms of the roots of the pure uni-serial Abelian quartic are obtained by two different methods (§ 27–§ 39). Then the necessary and sufficient forms of the roots of the pure uni-serial Abelian of a degree which is the continued product of any number of distinct prime numbers are found (§ 40–§ 46). Then the problem is solved for the pure uni-serial Abelian of a degree which is four times the continued product of any number of distinct odd

primes (§47–§57). Finally, from the relation between the solvable irreducible equation of prime degree n and the pure uni-serial Abelian equation of degree $n-1$, the necessary and sufficient forms of the roots of the irreducible solvable equation of prime degree n are shown to be determinable for all cases in which $n-1$ is either the continued product of a number of distinct primes, or four times the continued product of a number of distinct odd primes (§58–§64).

PRELIMINARY.

Corollary from a Law of Kronecker.

§4. It was proved by Kronecker that, n being any integer, the primitive n^{th} roots of unity are the roots of an irreducible equation, that is, of an irreducible equation with rational coefficients. We shall have occasion to make use of the following Corollary from this law: Let w and w' be two primitive n^{th} roots of unity, and let $F(w)$ be a rational function of w. Then, if $F(w)=0$, $F(w')=0$. For, by hypothesis,

$$F(w)=hw^s+h_1w^{s-1}+\text{ etc.}=0,$$

where h, h_1, etc., are rational. We assume s to be less than n, and h to be distinct from zero; therefore

$$h^{-1}\{F(w)\}=w^s+h^{-1}h_1w^{s-1}+\text{ etc.}=0.$$

Therefore w is a root of the equation $\phi(x)=x^s+h^{-1}h_1x^{s-1}+\text{ etc.}=0$. If $\psi(x)=0$ be the equation whose roots are the primitive n^{th} roots of unity, w is a root of the equation $\psi(x)=0$. Therefore the equations $\phi(x)=0$ and $\psi(x)=0$ have a root in common. But, by Kronecker's law, the equation $\psi(x)=0$ is irreducible. Therefore $\phi(x)$ is divisible by $\psi(x)$ without remainder. This implies that all the roots of the equation $\psi(x)=0$ are roots of the equation $\phi(x)=0$. Therefore $\phi(w')=0$. Therefore $F(w')=0$.

Principles established by Abel.

§5. Let $f(x)=0$ be a uni-serial Abelian equation of the n^{th} degree, and let its roots, in the order in which they circulate, be the terms in (1). It is known (see Serret's Cours d'Algèbre supérieure, Vol. II, page 500, third edition) that

$$x_1=R_0^{\frac{1}{n}}+R_1^{\frac{1}{n}}+R_2^{\frac{1}{n}}+\ldots+R_{n-1}^{\frac{1}{n}},$$

where R_1 is a rational function of the primitive n^{th} root of unity w and of the known quantities involved in the coefficients of θ; and, z being any integer, R_z is derived from R_1 by changing w into w^z. Putting

$$x_{s+1} = R_0^{\frac{1}{n}} + w^s R_1^{\frac{1}{n}} + w^{2s} R_2^{\frac{1}{n}} + \ldots + w^{(n-1)s} R_{n-1}^{\frac{1}{n}}, \tag{2}$$

the n roots of the equation $f(x) = 0$ are obtained by giving s in x_{s+1} successively the values $0, 1, 2, \ldots, n-1$. Therefore $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation; consequently, $R_0^{\frac{1}{n}}$ is rational. An equation of the type

$$(R_z R_1^{-z})^{\frac{1}{n}} = F(w) \tag{3}$$

subsists for every integral value of z, $F(w)$ being a rational function of w and of the known quantities involved in the coefficients of θ. As w may be any one of the primitive n^{th} roots of unity, if the general primitive n^{th} root of unity be w^e, we may suppose w in R_1 to be changed into w^e. The n roots of the equation $f(x) = 0$ will then be obtained by giving t, in the expression

$$R_0^{\frac{1}{n}} + w^t R_e^{\frac{1}{n}} + w^{2t} R_{2e}^{\frac{1}{n}} + \text{etc.} \tag{4}$$

successively the values $0, 1, 2, \ldots, n-1$. Abel's investigation shows that the form of the function $F(w)$ in (3) is independent of the particular primitive n^{th} root of unity denoted by w. Hence the change of w into w^e causes equation (3) to become

$$(R_{ez} R_e^{-z})^{\frac{1}{n}} = F(w^e), \tag{5}$$

the symbol F having the same meaning for every value of e.

Fundamental Element of the Root.

§6. Because R_0, R_2, etc., are derived from R_1 by changing w into w^0, w^2, etc., the root x_1 can be constructed when R_1 is given. We may therefore call R_1 the fundamental element of the root. Examples of the way in which the root is constructed from its fundamental element will present themselves in the course of the paper.

A Certain Rational Function of the Primitive n^{th} Root of Unity, n being an Odd Prime Number.

§7. Taking n an odd prime number, there is a certain rational function of the primitive n^{th} root of unity w, of which we shall have occasion to make

frequent use. It will be convenient to describe it here, and to point out two of its properties. Let

$$w,\ w^{\lambda},\ w^{\lambda^2},\ \ldots\ldots,\ w^{\lambda^{n-2}}, \tag{6}$$

be a cycle containing all the primitive n^{th} roots of unity. The number λ may be assumed to be less than n. With a view to convenience in printing, the indices of the powers of w in (6) may be written

$$1,\ \lambda,\ \alpha,\ \beta,\ \ldots\ldots,\ \delta,\ \varepsilon,\ \theta; \tag{7}$$

that is to say, $\alpha = \lambda^2$, $\beta = \lambda^3$, and so on. Take P_1 a rational function of w, and, z being any integer, let P_z be what P_1 becomes when w is changed into w^z. Then the function to which we desire to call attention is

$$P_1^{\theta} P_{\lambda}^{\varepsilon} P_{\alpha}^{\delta} \ldots\ldots P_{\delta}^{\alpha} P_{\varepsilon}^{\lambda} P_{\theta}. \tag{8}$$

The *subscripts* of the factors of the expression (8) are the terms in (7), while the *indices* are the terms in (7) in reverse order. The expression (8) may be denoted by the symbol ϕ_1. From ϕ_1, as expressed in (8), derive ϕ_z by changing w into w^z, z being any integer. Then

$$\left.\begin{array}{l}
\phi_1 = P_1^{\theta} P_{\lambda}^{\varepsilon} P_{\alpha}^{\delta} \ldots\ldots P_{\delta}^{\alpha} P_{\varepsilon}^{\lambda} P_{\theta} \\
\phi_{\lambda} = P_1 P_{\lambda}^{\theta} P_{\alpha}^{\varepsilon} \ldots\ldots \quad P_{\varepsilon}^{\alpha} P_{\theta}^{\lambda} \\
\phi_{\alpha} = P_1^{\lambda} P_{\lambda} P_{\alpha}^{\theta} \ldots\ldots \quad P_{\theta}^{\alpha} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\phi_{\theta} = P_1^{\varepsilon} P_{\lambda}^{\delta} \quad \ldots\ldots P_{\delta}^{\lambda} P\, P_{\theta}^{\theta}
\end{array}\right\} \tag{9}$$

The second of these equations is derived from the first by changing w into w^{λ}. This, since $\alpha = \lambda^2$ and $\beta = \lambda^3$, and so on, causes w^{λ} to become w^{α}, and w^{α} to become w^{β}, and so on. Hence it causes P_{λ} to become P_{α}, P_{α} to become P_{β}, and so on. Thus the second of equations (9) is obtained. The rest are obtained in a similar manner.

§ 8. One property which the function ϕ_1 possesses is that $\phi_0^{\frac{1}{n}}$ has a rational value. For

$$\phi_0 = P_0^{\theta} P_0^{\varepsilon} \ldots\ldots P_0^{\lambda} P_0 = P_0^{t},$$

where

$$t = 1 + \lambda + \lambda^2 + \ldots\ldots + \lambda^{n-2} = \frac{\lambda^{n-1} - 1}{\lambda - 1}.$$

Because (6) is a cycle of primitive n^{th} roots of unity, $\lambda^{n-1} - 1$ is a multiple of n. And, since λ is less than n, $\lambda - 1$ is not a multiple of n; therefore t is a multiple of n. Put $t = mn$; then

$$\phi_0 = (P_0^{m})^{n};$$

consequently, one of the values of $\phi_0^{\frac{1}{n}}$ is the rational quantity P_0^{m}.

§ 9. A second property of the function ϕ_1 is that an equation of the type

$$(\phi_z \phi_1^{-z})^{\frac{1}{n}} = F(w) \tag{10}$$

subsists for every integral value of z, $F(w)$ being a rational function of w. For, taking $z = \lambda$,

$$\phi_1^\lambda = P_1^{\lambda\theta} P_\lambda^{\lambda\epsilon} P_\alpha^{\lambda\delta} \ldots\ldots P_\delta^{\lambda\alpha} P_\epsilon^{\lambda^2} P_\theta^\lambda .$$

But $\lambda^2 = \alpha$, $\alpha\lambda = \beta$, ..., $\lambda\varepsilon = \theta$. And $\lambda\theta = \lambda^{n-1}$. Since $\lambda^{n-1} - 1$ is a multiple of n, put $\lambda^{n-1} - 1 = cn$. Then

$$\phi_1^\lambda = P_1^{cn} (P_1 P_\lambda^\theta \ldots\ldots P_\theta^\lambda).$$

Comparing this with the second of equations (9),

$$\phi_\lambda \phi_1^{-\lambda} = P_1^{-cn}.$$

Therefore

$$(\phi_\lambda \phi_1^{-\lambda})^{\frac{1}{n}} = w' P_1^{-c}, \tag{11}$$

w' being an n^{th} root of unity. In like manner, from the second and third of equations (9),

$$\phi_\alpha \phi_\lambda^{-\lambda} = P_\lambda^{-cn}.$$

Substitute here the value of ϕ_λ in (11). Then $\phi_\alpha \phi_1^{-\alpha} = (P_\lambda^{-c} P_1^{-\lambda c})^n$. Therefore

$$(\phi_\alpha \phi_1^{-\alpha})^{\frac{1}{n}} = w'' (P_\lambda^{-c} P_1^{-\lambda c}), \tag{12}$$

w'' being an n^{th} root of unity. The equations (11) and (12) are of the type (10). Therefore an equation of the type (10) subsists when z is equal either to λ or to α. In the same way we can go on to show that an equation of the type (10) subsists when z is equal to any of the terms in (7). Should $z = 0$, $\phi_z \phi_1^{-z} = \phi_0$. Therefore, by § 8, $\phi_z \phi_1^{-z} = P_0^{mn}$. Hence in this case also $(\phi_z \phi_1^{-z})^{\frac{1}{n}}$ is a rational function of w. Therefore, whether z be zero or one of the terms in the series (7), an equation of the type (10) subsists. This implies that an equation of the type (10) subsists for every integral value of z.

Criterion of Pure Uni-Serial Abelianism.

The Criterion Stated.

§ 10. A Criterion of pure uni-serial Abelianism may now be given. Let R_1 be a rational function of the primitive n^{th} root of unity w, and, z being any integer, let R_z be derived from R_1 by changing w into w^z. Then, if $R_0^{\frac{1}{n}}$ is rational, and if the terms $R_1^{\frac{1}{n}}$, $R_z^{\frac{1}{n}}$, etc., are such that an equation of the type

(3) subsists for every integral value of z, an equation (5), in which the symbol F has the same meaning as in (3), at the same time subsisting for every value of e prime to n, the n values of x_{s+1} in (2), obtained by giving s successively the values $0, 1, 2, \ldots, n-1$, are the roots of a pure uni-serial Abelian equation, provided always that the equation of the n^{th} degree, of which they can be shown to be the roots, is irreducible.

Proof of the Criterion.

§ 11. Here we assume that the conditions specified in § 10 are satisfied, and we have to show that the n values of x_{s+1} in (2), obtained by putting s successively equal to $0, 1, 2, \ldots, n-1$, are the roots of a pure uni-serial Abelian equation.

§ 12. We will first prove that the n values of the expression (4) obtained by giving t successively the n values $0, 1, 2, \ldots, n-1$, are the same, the order of the terms not being considered, as the n values of x_{s+1} in (2) obtained by giving s successively the values $0, 1, 2, \ldots, n-1$.

Because w^e is a primitive n^{th} root of unity, all the n^{th} roots of unity distinct from unity are contained in the series

$$w^e,\ w^{2e},\ w^{3e},\ \ldots,\ w^{(n-1)e}.$$

Therefore the two series

$$R_1,\ R_2,\ R_3,\ \ldots,\ R_{n-1},$$
$$R_e,\ R_{2e},\ R_{3e},\ \ldots,\ R_{(n-1)e},$$

are identical with one another, the order of the terms not being considered. Therefore, also, the two series

$$R_1^{\frac{1}{n}},\ R_2^{\frac{1}{n}},\ R_3^{\frac{1}{n}},\ \ldots,\ R_{n-1}^{\frac{1}{n}},$$
$$R_e^{\frac{1}{n}},\ R_{2e}^{\frac{1}{n}},\ R_{3e}^{\frac{1}{n}},\ \ldots,\ R_{(n-1)e}^{\frac{1}{n}},$$

are identical with one another, the order of the terms not being considered, it being understood that $R_e^{\frac{1}{n}}$, $R_{2e}^{\frac{1}{n}}$, etc., are the same n^{th} roots of R_e, R_{2e}, etc., or of R_1, R_2, etc., that are taken in the series $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc. Let the expression (4) be called x'_{t+1}. The separate members of the expression x_{s+1} are

$$R_0^{\frac{1}{n}},\ w^s R_1^{\frac{1}{n}},\ w^{2s} R_2^{\frac{1}{n}},\ \text{etc.} \tag{13}$$

Taking s with a definite value, let

$$es = bn + c,$$

where b and c are whole numbers, and c is less than n. Then, putting $t = c$, the separate members of the expression x'_{c+1} are

$$R_0^{\frac{1}{n}},\ w^c R_e^{\frac{1}{n}},\ w^{2c} R_{2e}^{\frac{1}{n}},\ \text{etc.} \tag{14}$$

Because $es = bn + c$, $w^c = w^{es}$. Therefore $w^c R_e^{\frac{1}{n}} = w^{es} R_e^{\frac{1}{n}}$; that is, the second term in (14) is equal to the $(e+1)^{\text{th}}$ term in (13). Again, if $2e = dn + v$, where d and v are whole numbers, and v is less than n, $R_{2e}^{\frac{1}{n}} = R_v^{\frac{1}{n}}$. Also, because $es = bn + c$, $w^{2c} = w^{2es}$. Therefore $w^{2c} R_{2e}^{\frac{1}{n}} = w^{2es} R_v^{\frac{1}{n}} = w^{vs} R_v^{\frac{1}{n}}$; that is, the third term in (14) is equal to the $(v+1)^{\text{th}}$ term in (13); and so on. Hence $x'_{c+1} = x_{s+1}$. Let now s and σ be two distinct values of s, both less than n; and let

$$x'_{c+1} = x_{s+1}, \text{ and } x'_{z+1} = x_{\sigma+1}.$$

By what has been proved, the numbers c and z are determined by the equations

$$es = bn + c,\ e\sigma = \beta n + z,$$

bn and βn being multiples of n. But, since s and σ are different, and e is prime to n, c and z must be different. Hence, as x_{s+1} runs through its n values, x_1, x_2, etc., x'_{t+1} must run through its n values, severally equal, in some order, to those of x_{s+1}.

§13. From (5),

$$\begin{aligned} R_{2e}^{\frac{1}{n}} &= A_e R_e^{\frac{2}{n}}, \\ R_{3e}^{\frac{1}{n}} &= B_e R_e^{\frac{3}{n}}, \\ &\cdots\cdots\cdots \\ R_{(n-1)e}^{\frac{1}{n}} &= C_e R_e^{\frac{n-1}{n}}, \end{aligned}$$

where A_e, B_e, etc., are rational functions of w^e. These values of $R_{2e}^{\frac{1}{n}}$, $R_{3e}^{\frac{1}{n}}$, etc., substituted in (4), cause that expression to become

$$R_0^{\frac{1}{n}} + w^t R_e^{\frac{1}{n}} + w^{2t} A_e R_e^{\frac{2}{n}} + w^{3t} B_e R_e^{\frac{3}{n}} + \text{etc.} \tag{15}$$

Let the n values of the expression (15), obtained by putting t successively equal to $0, 1, 2, \ldots, n-1$, be

$$r_1,\ r_2,\ \ldots,\ r_n. \tag{16}$$

Then, v being a whole number,

$$\begin{aligned} r_1^v &= a_e + b_e R_e^{\frac{1}{n}} + c_e R_e^{\frac{2}{n}} + \ldots + d_e R_e^{\frac{n-1}{n}}, \\ r_2^v &= a_e + w b_e R_e^{\frac{1}{n}} + w^2 c_e R_e^{\frac{2}{n}} + \ldots + w^{(n-1)} d_e R_e^{\frac{n-1}{n}}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ r_n^v &= a_e + w^{-1} b_e R_e^{\frac{1}{n}} + w^{-2} c_e R_e^{\frac{2}{n}} + \ldots + w d_e R_e^{\frac{n-1}{n}}, \end{aligned}$$

where a_e, b_e, etc., are rational functions of w^e. Therefore, if S_v be the sum of the v^{th} powers of the terms in (16), $S_v = na_e$. Because a_e is a rational function of w^e, we may put

$$na_e = g + hw^e + kw^{2e} + \dots + lw^{(n-1)e}, \text{ where } g,\ h, \text{ etc., are rational.}$$

But, by § 12, the n values of the expression (15), obtained by giving t successively the values $0, 1, 2, \dots, n-1$, are the same whatever value, making w^e a primitive n^{th} root of unity, be given to e. We may therefore substitute for w^e, in the expression for na_e or S_v, any one of the primitive n^{th} roots of unity

$$w,\ w^c,\ w^d,\ \dots,\ w^t. \tag{17}$$

Therefore

$$\begin{aligned} S_v &= g + hw + kw^2 + \text{etc.} \\ &= g + hw^c + kw^{2c} + \text{etc.} \\ &\dots\dots\dots\dots\dots \\ &= g + hw^t + kw^{2t} + \text{etc.} \end{aligned}$$

Therefore

$$mS_v = mg + h(w + w^c + \text{etc.}) + k(w^2 + w^{2c} + \text{etc.}) + \text{etc.},$$

m being the number of the terms in the series (17). Consequently S_v is a rational and symmetrical function of the primitive n^{th} roots of unity. Hence, by Kronecker's law, referred to in § 4, S_v is rational. This implies that the n terms in (16), which have been shown to be identical with the n values of x_{s+1} in (2) obtained by giving s successively the values $0, 1, 2, \dots, n-1$, are the roots of an equation of the n^{th} degree; that is, of an equation of the n^{th} degree with rational coefficients. Let this equation be $f(x) = 0$.

§ 14. In accordance with the proviso in § 10, let the equation $f(x) = 0$ be irreducible. It is then a pure Abelian. For, taking r_1, r_2, etc., as in § 13,

$$\left.\begin{aligned} r_1 &= R_0^{\frac{1}{n}} + R_e^{\frac{1}{n}} + A_e R_e^{\frac{2}{n}} + \dots + C_e R_e^{\frac{n-1}{n}} \\ r_1^2 &= D_e + F_e R_e^{\frac{1}{n}} + G_e R_e^{\frac{2}{n}} + \dots + H_e R_e^{\frac{n-1}{n}} \\ &\dots\dots\dots\dots\dots\dots\dots \\ r_1^{n-1} &= K_e + L_e R_e^{\frac{1}{n}} + M_e R_e^{\frac{2}{n}} + \dots + Q_e R_e^{\frac{n-1}{n}} \end{aligned}\right\} \tag{18}$$

where A_e, D_e, F_e, etc., are rational functions of w^e. Multiply the first of equations (18) by h_e, the second by k_e, and so on, the last being multiplied by l_e; then, by addition,

$$\begin{aligned} h_e r_1 + k_e r_1^2 + \dots + l_e r_1^{n-1} = {} & \left(h_e R_0^{\frac{1}{n}} + k_e D_e + \dots + l_e K_e\right) \\ & + (h_e + k_e F_e + \dots + l_e L_e) R_e^{\frac{1}{n}} \\ & + \dots\dots\dots\dots\dots \\ & + (h_e C_e + k_e H_e + \dots + l_e Q_e) R_e^{\frac{n-1}{n}}. \end{aligned}$$

Let the $n-1$ quantities, h_e, k_e, etc., be determined by the $n-1$ equations

$$\begin{aligned} &h_e \quad + k_e F_e + \ldots . + l_e L_e = w^e, \\ &h_e A_e + k_e G_e + \ldots . + l_e M_e = w^{2e} A_e, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &h_e C_e + k_e H_e + \ldots . + l^e Q_e = w^{(n-1)e} C_e. \end{aligned}$$

Then $\quad h_e r_1 + k_e r_1^2 + \text{etc.} = \left(h_e R_0^{\frac{1}{n}} + k_e D_e + \ldots . + l_e K_e\right)$

$$+ w^e R_e^{\frac{1}{n}} + w^{2e} A_e R_e^{\frac{2}{n}} + \ldots . + w^{(n-1)e} C_e R_e^{\frac{n-1}{n}};$$

or, putting $R_{2e}^{\frac{1}{n}}$ for $A_e R_e^{\frac{2}{n}}$, and so on,

$$h_e r_1 + k_e r_1^2 + \text{etc.} = \left(h_e R_0^{\frac{1}{n}} + k_e D_e + \ldots . + l_e K_e\right)$$
$$+ w^e R_e^{\frac{1}{n}} + w^{2e} R_{2e}^{\frac{1}{n}} + \ldots . + w^{(n-1)e} R_{(n-1)e}^{\frac{1}{n}}. \qquad (19)$$

By § 12, $x'_{e+1} = x_{s+1}$, where $es = bn + c$. When $s = 0$, $c = 0$, and when $s = 1$, $c = e$; therefore

$$x_1 = x'_1 = R_0^{\frac{1}{n}} + R_e^{\frac{1}{n}} + R_{2e}^{\frac{1}{n}} + \text{etc.} = r_1,$$

and $\quad x_2 = x'_{e+1} = R_0^{\frac{1}{n}} + w^e R_e^{\frac{1}{n}} + w^{2e} R_{2e}^{\frac{1}{n}} + \text{etc.}$

Therefore (19) may be written

$$h_e x_1 + k_e x_1^2 + \text{etc.} = \left(h_e R_0^{\frac{1}{n}} + \text{etc.}\right) - R_0^{\frac{1}{n}} + x_2.$$

But e may be any number that makes w^e a primitive n^{th} root of unity, and (17) is the series of the primitive n^{th} roots of unity. Therefore

$$\begin{aligned} x_2 &= \left\{R_0^{\frac{1}{n}} - \left(h_1 R_0^{\frac{1}{n}} + \text{etc.}\right)\right\} + h_1 x_1 + k_1 x_1^2 + \ldots . + l_1 x_1^{n-1} \\ &= \left\{R_0^{\frac{1}{n}} - \left(h_c R_0^{\frac{1}{n}} + \text{etc.}\right)\right\} + h_c x_1 + k_c x_1^2 + \ldots . + l_c x_1^{n-1} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \left\{R_0^{\frac{1}{n}} - \left(h_e R_0^{\frac{1}{n}} + \text{etc.}\right)\right\} + h_e x_1 + k_e x_1^2 + \ldots . + l_e x_1^{n-1}, \end{aligned}$$

where h_1, h_c, etc., are what h_e becomes when w^e is changed into w, w^c, etc., and k_1, k_c, etc., are what k_e becomes when w^e is changed into w, w^c, etc., and so on. Therefore, by addition, m being the number of the primitive n^{th} roots of unity,

$$m x_2 = p + q x_1 + t x_1^2 + \ldots . + v x_1^{n-1},$$

where p, q, etc., are rational and symmetrical functions of the primitive n^{th} roots of unity, and therefore are rational. Hence x_2 is a rational function of x_1. Therefore the equation $f(x) = 0$ is a pure Abelian.

§ 15. It is also uni-serial. For, by what has been proved,

$$x_2 = \theta x_1,$$

θx_1 denoting a rational function of x_1. But, from the form of x_{s+1} in (2), since $R_2^{\frac{1}{n}} = A_1 R_1^{\frac{2}{n}}$, and $R_3^{\frac{1}{n}} = B_1 R_1^{\frac{3}{n}}$, and so on, we pass from x_1 to x_2 by simply changing $R_1^{\frac{1}{n}}$ into $wR_1^{\frac{1}{n}}$. The same change transforms x_2 into x_3. Therefore

$$x_3 = \theta x_2 = \theta^2 x_1.$$

In like manner $x_4 = \theta^3 x_1$, and so on, till ultimately $\theta^n x_1 = x_1$. Thus all the roots of the equation $f(x) = 0$ are comprised in the series

$$x_1,\ \theta x_1,\ \theta^2 x_1,\ \ldots\ldots,\ \theta^{n-1} x_1.$$

PURE ABELIAN EQUATIONS OF ODD PRIME DEGREES.

Fundamental Element of the Root; the Root Constructed from its Fundamental Element.

§ 16. We confine ourselves to pure Abelians of *odd* prime degrees, because the irreducible quadratic is always a pure Abelian. Let n be an odd prime number, and let the primitive n^{th} roots of unity be the terms w, w^{λ}, w^{λ^2}, etc., forming the series (6). Take ϕ_1 as in the first of equations (9); then, if R_1 be the fundamental element (see § 6) of the root of a pure Abelian equation $f(x) = 0$ of the n^{th} degree, it will be found that

$$R_1 = A_1^n \phi_1, \tag{20}$$

A_1 being a rational function of w.

§ 17. From R_1, as expressed in (20), derive R_0, R_2, etc., by changing w into w^0, w^2, etc. By § 5, the root of the equation $f(x) = 0$ is

$$R_0^{\frac{1}{n}} + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \ldots\ldots + R_{n-1}^{\frac{1}{n}}. \tag{21}$$

To construct the root, we have to determine the particular n^{th} roots of R_0, R_1, etc., that are to be taken together in (21). When w is changed into w^s, let A_1 become A_s, as ϕ_1 becomes ϕ_s. Then

$$R_s = A_s^n \phi_s.$$

Therefore
$$R_s^{\frac{1}{n}} = w' A_s \phi_s^{\frac{1}{n}}, \tag{22}$$

w' being an n^{th} root of unity. In proceeding to make $R_s^{\frac{1}{n}}$ definite, we may first make $\phi_s^{\frac{1}{n}}$ definite. By (9),

$$\phi_1^{\frac{1}{n}} = w^a \left(P_1^{\theta} P_{\lambda}^{\epsilon} P_{\kappa}^{\delta} \ldots\ldots P_{\theta}\right)^{\frac{1}{n}},$$

w^a being an n^{th} root of unity. Let

$$P_1^{\frac{1}{n}},\ P_\lambda^{\frac{1}{n}},\ P_a^{\frac{1}{n}},\ \ldots\ldots,\ P_\theta^{\frac{1}{n}}, \tag{23}$$

be determinate; then, by taking w^a with the value unity, we get $\phi_1^{\frac{1}{n}}$ with the determinate value $\qquad \phi_1^{\frac{1}{n}} = (P_1^\theta P_\lambda^\epsilon P_a^\delta \ldots\ldots P_\theta)^{\frac{1}{n}}.$

Let us now consider $\phi_\lambda^{\frac{1}{n}}$. By (9), w^c being an n^{th} root of unity,

$$\phi_\lambda^{\frac{1}{n}} = w^c (P_1 P_\lambda^\theta P_a^\epsilon \ldots\ldots P_\theta^\lambda)^{\frac{1}{n}}.$$

Understanding that $P_1^{\frac{1}{n}}$, $P_\lambda^{\frac{1}{n}}$, etc., on the right-hand side of this equation are the same quantities that appear in (23), they have already been made definite. We can then make $\phi_\lambda^{\frac{1}{n}}$ definite by taking w^c with the value unity. Generally, if z be any number in the series $1,\ 2,\ \ldots\ldots,\ n-1$,

$$\phi_z^{\frac{1}{n}} = w^d (P_z^\theta P_{z\lambda}^\epsilon P_{za}^\delta \ldots\ldots P_{z\theta})^{\frac{1}{n}},$$

w^d being an n^{th} root of unity. Because z is prime to n, the $n-1$ terms $w^z,\ w^{z\lambda},\ w^{za},\ \ldots\ldots,\ w^{z\theta}$, are the same, in a certain order, with the terms $w,\ w^\lambda,\ w^a,\ \ldots\ldots,\ w^\theta$. Therefore the terms

$$P_z^{\frac{1}{n}},\ P_{z\lambda}^{\frac{1}{n}},\ P_{za}^{\frac{1}{n}},\ \ldots\ldots,\ P_{z\theta}^{\frac{1}{n}},$$

may be taken to be the same, in a certain order, with the terms in (23). They are accordingly determinate. We may then make $\phi_z^{\frac{1}{n}}$ definite by taking w^d with the value unity. Therefore, for every value of z in the series $1,\ 2,\ \ldots\ldots,\ n-1$,

$$\phi_z^{\frac{1}{n}} = (P_z^\theta P_{z\lambda}^\epsilon P_{za}^\delta \ldots\ldots P_{z\theta})^{\frac{1}{n}}. \tag{24}$$

Having thus determined $\phi_z^{\frac{1}{n}}$, we can make $R_z^{\frac{1}{n}}$ definite by taking w' in (22) equal to unity for every value of z in the series $1,\ 2,\ \ldots\ldots,\ n-1$; that is,

$$\left.\begin{array}{l} R_1^{\frac{1}{n}} = A_1 \phi_1^{\frac{1}{n}} \\ R_\lambda^{\frac{1}{n}} = A_\lambda \phi_\lambda^{\frac{1}{n}} \\ \ldots\ldots\ldots \\ R_\theta^{\frac{1}{n}} = A_\theta \phi_\theta^{\frac{1}{n}} \end{array}\right\} \tag{25}$$

As regards $R_0^{\frac{1}{n}}$, we have $R_0 = A_0^n \phi_0$. But, by § 8, $\phi_0 = P_0^{mn}$. Therefore $\phi_0^{\frac{1}{n}}$ has a rational value. Consequently $R_0^{\frac{1}{n}}$ has a rational value. In (21) substitute the rational value of $R_0^{\frac{1}{n}}$, and the values of $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, etc., given in (25), and the

root is constructed. In other words, the expression (21) is the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it can be shown to be the root, is irreducible.

Necessity of the above Forms.

§ 18. The root x_1 of the pure Abelian equation $f(x) = 0$ of the n^{th} degree, n an odd prime, being assumed to be expressible as in (21), we have to show that its fundamental element R_1 has the form (20), and that $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, etc., are to be taken as in (25), while $R_0^{\frac{1}{n}}$ receives its rational value.

§ 19. By (3), z being any integer,

$$R_z^{\frac{1}{n}} = \{F(w)\}\, R_1^{\frac{z}{n}},$$

$F(w)$ being a rational function of w. And equation (5) subsists along with (3); that is, e being any whole number prime to n,

$$R_{ez}^{\frac{1}{n}} = \{F(w^e)\}\, R_e^{\frac{z}{n}}.$$

Give z here successively the values $1, \lambda, \alpha$, etc., these terms being the same as in the series (7). Then

$$\begin{aligned} R_e^{\frac{1}{n}} &= R_e^{\frac{1}{n}}, \\ R_{e\lambda}^{\frac{1}{n}} &= B_e R_e^{\frac{\lambda}{n}}, \\ R_{e\alpha}^{\frac{1}{n}} &= C_e R_e^{\frac{\alpha}{n}}, \\ &\cdots\cdots\cdots \\ R_{e\theta}^{\frac{1}{n}} &= D_e R_e^{\frac{\theta}{n}}, \end{aligned}$$

B_e, C_e, etc., being rational functions of w^e. Therefore

$$(R_e^\theta R_{e\lambda}^\varepsilon R_{e\alpha}^\delta \ldots\ldots R_{e\theta})^{\frac{1}{n}} = G_e R_e^{\frac{t}{n}},$$

where G_e is a rational function of w^e, and

$$t = \theta + \varepsilon\lambda + \delta\alpha + \ldots\ldots + \theta.$$

From the nature of the series (7), $\theta = \lambda^{n-2}$, and $\varepsilon = \lambda^{n-3}$. Therefore $\varepsilon\lambda = \theta$. In like manner, each of the $n - 1$ separate members of t is equal to θ. Therefore $t = (n-1)\theta$. Because (6) is a cycle of primitive n^{th} roots of unity, in other words, because λ is a prime root of n, and $\theta = \lambda^{n-2}$, θ is prime to n. And $n - 1$ is necessarily prime to n. Therefore whole numbers h and k exist such that

$$ht = kn + 1.$$

Therefore
$$(R_e^\theta R_{e\lambda}^\epsilon \dots . R_{e\theta})^{\frac{h}{n}} = (G_e^h R_e^k) R_e^{\frac{1}{n}}.$$

For every integral value of z let $(R_{ez}^h)^{\frac{1}{n}}$ be written $P_{ez}^{\frac{1}{n}}$; then, putting A_e^{-1} for $G_e^h R_e^k$,

$$R_e^{\frac{1}{n}} = A_e (P_e^\theta P_{e\lambda}^\epsilon P_{ea}^\delta \dots . P_{e\theta})^{\frac{1}{n}}. \tag{26}$$

Hence, by putting $e = 1$, and taking ϕ_1 as in (9),

$$R_1 = A_1^n \phi_1.$$

Thus the form of the fundamental element in (20) is established. Also, when $e = 1$,

$$R_1^{\frac{1}{n}} = A_1 (P_1^\theta P_\lambda^\epsilon P_a^\delta \dots . P_\theta)^{\frac{1}{n}}.$$

Therefore, by (24), $R_1^{\frac{1}{n}} = A_1 \phi_1^{\frac{1}{n}}$. This is the first of equations (25). Since e may be any term prime to n, let $e = \lambda$. Then, from (26), because $\lambda^2 = \alpha$ and $\lambda\alpha = \beta$, and so on,

$$R_\lambda^{\frac{1}{n}} = A_\lambda (P_\lambda^\theta P_a^\epsilon P_\beta^\delta \dots . P_1)^{\frac{1}{n}}.$$

Therefore, giving z in (24) the value λ, $R_\lambda^{\frac{1}{n}} = A_\lambda \phi_\lambda^{\frac{1}{n}}$. This is the second of equations (25). In like manner we can show that all the terms $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, $\dots .$, $R_\theta^{\frac{1}{n}}$ are to be taken as in (25). It has only to be added that $R_0^{\frac{1}{n}}$ must be taken with its rational value, because, by § 5, $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation $f(x) = 0$.

Sufficiency of the Forms.

§ 20. We here assume that R_1 has the form (20), that $R_0^{\frac{1}{n}}$ is rational, and that $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, etc., are taken as in (25), and we have to show that the n values of x_{s+1} in (2), obtained by giving s successively the n values $0, 1, 2, \dots ., n-1$, are the roots of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which they are the roots, is irreducible. *In the first place,* $R_0^{\frac{1}{n}}$ has been taken rational. *In the next place,* an equation of the type (3) subsists for every integral value of z. For, let z not be a multiple of n. In this case it may be taken to be a number in the series $1, 2, \dots ., n-1$. Then, by (25),

$$(R_z R_1^{-z})^{\frac{1}{n}} = (A_z A_1^{-z})(\phi_z \phi_1^{-z})^{\frac{1}{n}}. \tag{27}$$

But ϕ_1 is the expression (8). Therefore, by § 9,

$$(\phi_z \phi_1^{-z})^{\frac{1}{n}} = F(w),$$

$F(w)$ being a rational function of w. This makes (27) an equation of the type (3). Next, let z be a multiple of n, in which case it may be taken to be zero. Then

$$R_z^{\frac{1}{n}} = R_0^{\frac{1}{n}}, \text{ and } R_1^{\frac{z}{n}} = 1.$$

Therefore

$$(R_z R_1^{-z})^{\frac{1}{n}} = R_0^{\frac{1}{n}}. \qquad (28)$$

Since $R_0^{\frac{1}{n}}$ is rational, (28) is an equation of the type (3). Therefore, whether z be a multiple of n or not, an equation of the type (3) subsists. *In the third place*, the equation (5) subsists along with (3) for every value of e that makes w^e a primitive n^{th} root of unity. For, let z be a multiple of n; it may be taken to be zero. Therefore

$$R_{ez}^{\frac{1}{n}} = R_0^{\frac{1}{n}}, \text{ and } R_e^{\frac{z}{n}} = 1.$$

Therefore

$$(R_{ez} R_e^{-z})^{\frac{1}{n}} = R_0^{\frac{1}{n}}. \qquad (29)$$

But, equation (28) being regarded as (3), (29) is (5). Next, let z not be a multiple of n. It may be taken to be a number in the series $1, 2, \ldots\ldots, n-1$. Then equation (27) is (3). But, in (27), z may be any number not a multiple of n, and ez is not a multiple of n. Therefore we may substitute for z either ez or e. Thus we have

$$(R_{ez} R_1^{-ez})^{\frac{1}{n}} = (A_{ez} A_1^{-ez})(\phi_{ez}\phi_1^{-ez})^{\frac{1}{n}}$$

and

$$(R_e R_1^{-e})^{\frac{1}{n}} = (A_e A_1^{-e})(\phi_e \phi_1^{-e})^{\frac{1}{n}}.$$

Therefore

$$(R_{ez} R_e^{-z})^{\frac{1}{n}} = (A_{ez} A_e^{-z})(\phi_{ez}\phi_e^{-z})^{\frac{1}{n}}. \qquad (30)$$

But, equation (27) being regarded as (3), equation (30) is (5). Therefore, whether z be a multiple of n or not, equation (5) subsists along with (3). Hence, by the Criterion in § 10, the n values of x_{s+1} in (2), obtained by giving s successively the values $0, 1, 2, \ldots\ldots, n-1$, are the roots of a pure uni-serial Abelian equation.

Particular Values of n; the Pure Abelian Cubic.

§ 21. When the equation $f(x) = 0$ is of the third degree, taking $\lambda = 2$, the series (7) is reduced to the terms 1, 2, and the equations (25) become

$$R_1^{\frac{1}{3}} = A_1 (P_1^2 P_2)^{\frac{1}{3}}, \quad R_2^{\frac{1}{3}} = A_2 (P_2^2 P_1)^{\frac{1}{3}}.$$

Also $R_0^{\frac{1}{3}} = A_0 \phi_0$. Therefore

$$x_1 = A_0\phi_0 + A_1 (P_1^2 P_2)^{\frac{1}{3}} + A_2 (P_1 P_2^2)^{\frac{1}{3}}.$$

If $A_0\phi_0 = 0$, the equation wants its second term. Then, putting

$$\psi_1 = A_1^2 A_2^{-1} P_1 \text{ and } \psi_2 = A_2^2 A_1^{-1} P_2,$$

we get

$$x_1 = (\psi_1^2\psi_2)^{\frac{1}{3}} + (\psi_2^2\psi_1)^{\frac{1}{3}}.$$

§ 22. Let the pure Abelian cubic of which x_1 is the root be

$$x^3 + px + q = 0.$$

Because ψ_1 is a rational function of the primitive third root of unity,

$$\psi_1 = b + c\sqrt{-3}$$

and

$$\psi_2 = b - c\sqrt{-3},$$

b and c being rational. Therefore $\psi_1\psi_2 = b^2 + 3c^2$. Therefore

$$x_1 = \{(b^2+3c^2)(b+c\sqrt{-3})\}^{\frac{1}{3}} + \{(b^2+3c^2)(b-c\sqrt{-3})\}^{\frac{1}{3}}.$$

But

$$x_1 = \left\{-\frac{q}{2} + \sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}\right\}^{\frac{1}{3}} + \left\{-\frac{q}{2} - \sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}\right\}^{\frac{1}{3}}.$$

Therefore

$$\sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)} = c\,(b^2+3c^2)\sqrt{-3}.$$

Therefore

$$\sqrt{(-4p^3-27q^2)} = 18c\,(b^2+3c^2).$$

Thus $\sqrt{(-4p^3-27q^2)}$ is rational: the well known relation between the coefficients which makes the irreducible cubic $x^3+px+q=0$ a pure Abelian.

The Pure Abelian Quintic.

§ 23. When $n = 5$, λ may be taken to be 2. The series (7) then becomes 1, 2, 4, 8; or, rejecting multiples of 5, 1, 2, 4, 3. We may then put

$$R_1^{\frac{1}{5}} = A_1(P_1^3P_2^4P_4^2P_3)^{\frac{1}{5}},$$
$$R_2^{\frac{1}{5}} = A_2(P_1P_2^3P_4^4P_3^2)^{\frac{1}{5}},$$
$$R_4^{\frac{1}{5}} = A_4(P_1^2P_2P_4^3P_3^4)^{\frac{1}{5}},$$
$$R_1^{\frac{1}{5}} = A_3(P_1^4P_2^2P_4P_3^3)^{\frac{1}{5}}.$$

If we assume R_0 to be zero,

$$x_1 = A_1(P_1^3P_2^4P_4^2P_3)^{\frac{1}{5}} + A_2(P_1P_2^3P_4^4P_3^2)^{\frac{1}{5}} + A_4(P_1^2P_2P_4^3P_3^4)^{\frac{1}{5}} + A_3(P_1^4P_2^2P_4P_3^3)^{\frac{1}{5}}. \quad (31)$$

§ 24. In a celebrated fragment (see Crelle's Journal, Vol. V, p. 336) found among the papers of Abel after his death, the root r_1 of the solvable equation of the fifth degree wanting the second term was stated, though without any accompanying demonstration, substantially as follows: Let

$$\left.\begin{aligned} a_1 &= p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})} \\ a_2 &= p - q\sqrt{z} + \sqrt{(hz - h\sqrt{z})} \\ a_4 &= p + q\sqrt{z} - \sqrt{(hz + h\sqrt{z})} \\ a_3 &= p - q\sqrt{z} - \sqrt{(hz - h\sqrt{z})} \end{aligned}\right\} \quad (32)$$

where p, q and h are rational, and

$$z = e^2 + 1, \tag{33}$$

e being rational. Then, B_1 being a rational function of α_1, B_2 the same rational function of α_2, and so on,

$$r_1 = B_1\,(\alpha_1^3\alpha_2^4\alpha_4^2\alpha_3)^{\frac{1}{5}} + B_2(\alpha_1\alpha_2^3\alpha_4^4\alpha_3^2)^{\frac{1}{5}} + B_4(\alpha_1^2\alpha_2\alpha_4^3\alpha_3^4)^{\frac{1}{5}} + B_3(\alpha_1^4\alpha_2^2\alpha_4\alpha_3^3)^{\frac{1}{5}}. \tag{34}$$

§ 25. The expression for r_1 in (34) is the root of a solvable irreducible quintic, not necessarily a pure Abelian. To obtain from it the necessary and sufficient form of the root of a pure Abelian quintic, we make use of the law referred to in § 5, according to which the root of the pure Abelian quintic wanting the second term is

$$R_1^{\frac{1}{5}} + R_2^{\frac{1}{5}} + R_3^{\frac{1}{5}} + R_4^{\frac{1}{5}},$$

where R_1 is a rational function of the primitive fifth root of unity w. By this law, to deduce the root x_1 of a pure Abelian quintic from the root r_1 of an irreducible solvable quintic as in (34), we have simply to pass from the more general expression α_1 to the less general expression which we have called P_1, because, in doing this, we necessarily pass from B_1 to A_1, B_1 being a rational function of α_1, and A_1 a rational function of P_1. The question, however, is: Can we pass from α_1 to P_1? In other words, can the general rational function of the primitive fifth root of unity be subsumed under α_1? That it can, may be thus shown: The value of w is

$$w = \frac{\sqrt{5}-1}{4} + \frac{\sqrt{(-10-2\sqrt{5})}}{4}. \tag{35}$$

Hence, if $F(w)$ be the general rational function of w,

$$F(w) = p + k\sqrt{5} + (l + m\sqrt{5})\sqrt{(-10-2\sqrt{5})}, \tag{36}$$

where p, k, l and m are rational. Putting

$$z = \frac{5\,(l^2 + 5m^2 + 2lm)^2}{(l^2 + 5m^2 + 10lm)^2}$$

and

$$h = \frac{-2\,(l^2 + 5m^2 + 10lm)^2}{l^2 + 5m^2 + 2lm},$$

(36) becomes

$$F(w) = p + \frac{k\,(l^2 + 5m^2 + 10lm)\sqrt{z}}{l^2 + 5m^2 + 2lm} + \sqrt{(hz + h\sqrt{z})};$$

or, putting

$$q = \frac{k\,(l^2 + 5m^2 + 10lm)}{l^2 + 5m^2 + 2lm},$$

$$F(w) = p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})}. \tag{37}$$

The value of z given above conforms to the type (33), for it can be changed into

$$z=\left\{\frac{2(l^2-5m^2)}{l^2+5m^2+10lm}\right\}^2+1.$$

Hence the general rational function of the primitive fifth root of unity falls under the expression for α_1 in (32).

§ 26. The writer may perhaps be permitted to refer to a paper of his, entitled "Solution of Solvable Irreducible Quintic Equations," which appeared in this Journal, Vol. VII, No. 2. Assuming that the quintic to be solved has, by Jerrard's application of the method of Tschirnhaus, been brought to the trinomial form

$$x^5+px+q=0, \tag{38}$$

he proved, in the article referred to, that it admits of algebraical solution only if

$$p=\frac{5A^4(3-B)}{16+B^2}$$

and

$$q=\frac{A^5(22+B)}{16+B^2}.$$

When the coefficients are thus related, take λ a root of the equation

$$x^4-Bx^3-6x^2+Bx+1=0.$$

Put

$$a=\frac{-(\lambda^2+1)}{A\lambda(\lambda-1)}$$

and

$$\theta=\frac{-A^5\lambda(\lambda-1)^3}{(16+B^2)(\lambda+1)(\lambda^2+1)};$$

then the solution of the equation (38) is

$$r_1=\theta^{\frac{1}{5}}+a\theta^{\frac{2}{5}}+\lambda a^2\theta^{\frac{3}{5}}-\lambda a^3\theta^{\frac{4}{5}}.$$

This form of the root may at first sight seem to have no affinity with the Abelian form in (34); but, in a communication which was laid before the Royal Society of Canada at its meeting in May, 1886, and which is to appear in the forthcoming volume of the Transactions of the Society, the writer has shown the essential identity of the two forms.

The Pure Uni-Serial Abelian Quartic.

Necessary and Sufficient Forms of the Roots.

§ 27. Taking $z=e^2+1$ as in (33), the necessary and sufficient forms of the roots of the pure uni-serial Abelian quartic are the expressions α_1, α_2, α_4, α_3 in

(32); the rational expressions p, q, h, e being subject to the sole restriction that they must leave the equation of the fourth degree, which has α_1, α_2, α_4 and α_3 for its roots, irreducible. There is thus an intimate relation between the pure uni-serial Abelian of the fourth degree and the solvable irreducible equation of the fifth degree. This is only a case of a more general law. If $2n+1$ be any prime number, and if the forms of the roots of the pure uni-serial Abelian of degree $2n$ have been found, the necessary and sufficient forms of the roots of the solvable irreducible equation of degree $2n+1$ can be found.

Necessity of the Forms (32).

§ 28. Here an equation of the fourth degree $f(x)=0$ is assumed to be a pure uni-serial Abelian; and we have to show that its roots are of the forms α_1, α_2, α_4, α_3 in (32). The roots of the equation $f(x)=0$, in the familiar Abelian notation, are

$$x_1,\ \theta x_1,\ \theta^2 x_1,\ \theta^3 x_1. \tag{39}$$

Because x_1 is the root of an irreducible quartic, its form is

$$x_1 = P + \sqrt{Q},$$

where P is clear of the radical $\sqrt{Q}$. Another root of the quartic is $P-\sqrt{Q}$. This is obtained from x_1 by changing the sign of $\sqrt{Q}$; and, by changing the sign of $\sqrt{Q}$ in $P-\sqrt{Q}$, we return to $P+\sqrt{Q}$ or x_1. Hence $P-\sqrt{Q}$ must be the third term in (39). Therefore

$$\theta^2 x_1 = P - \sqrt{Q}.$$

In passing from x_1 to θx_1, let P and Q become P' and Q' respectively; then

$$\theta x_1 = P' + \sqrt{Q'};$$

therefore

$$\theta^3 x_1 = P' - \sqrt{Q'}.$$

In running through the series (39), the root of the equation $f(x)=0$ undergoes all its possible changes. But, from the expressions that have been obtained for x_1, θx_1, $\theta^2 x_1$ and $\theta^3 x_1$, P can take only the two values P, P', and Q can take only the two values Q, Q'. Therefore each of the expressions P and Q is the root of a quadratic equation. Hence the only radicals occurring in x_1 are square roots. But, when square roots are the only radicals in the root of an equation of the fourth degree, its root must be either

$$\left.\begin{array}{l} p+\sqrt{s}+\sqrt{t}, \\ \text{or}\quad p+k\sqrt{s}+\sqrt{(l+m\sqrt{s})} \end{array}\right\} \tag{40}$$

p, s, t, k, l and m being rational. Suppose, if possible, that x_1 is of the first of the forms (40); then either

$$\theta x_1 = p + \sqrt{s} - \sqrt{t} \therefore \theta^2 x_1 = p + \sqrt{s} + \sqrt{t} = x_1,$$

or

$$\theta x_1 = p - \sqrt{s} + \sqrt{t} \therefore \theta^2 x_1 = p + \sqrt{s} + \sqrt{t} = x_1,$$

or

$$\theta x_1 = p - \sqrt{s} - \sqrt{t} \therefore \theta^2 x_1 = p + \sqrt{s} + \sqrt{t} = x_1.$$

But the equation $f(x) = 0$, being a pure Abelian, is irreducible, and therefore cannot have equal roots. Therefore x_1 is not of the first of the forms (40). It is therefore of the second. Consequently we may put

$$\left.\begin{aligned} x_1 &= p + k\sqrt{s} + \sqrt{(l + m\sqrt{s})} \\ \theta x_1 &= p - k\sqrt{s} + \sqrt{(l - m\sqrt{s})} \\ \theta^2 x_1 &= p + k\sqrt{s} - \sqrt{(l + m\sqrt{s})} \\ \theta^3 x_1 &= p - k\sqrt{s} - \sqrt{(l - m\sqrt{s})} \end{aligned}\right\} \tag{41}$$

It is plain that $\theta^2 x_1$ must have the place assigned to it in (41), because the change that causes x_1 to become $\theta^2 x_1$ must transform $\theta^2 x_1$ into x_1. We can now determine the expression $\sqrt{(l + m\sqrt{s})}$ more definitely. To pass from x_1 to θx_1 we change the sign of $\sqrt{s}$ and take the resulting radical $\sqrt{(l - m\sqrt{s})}$ with the positive sign. In order that these changes may cause θx_1 to become $\theta^2 x_1$, the changes must admit of being made on θx_1. In other words, the radical $\sqrt{(l - m\sqrt{s})}$, which does not occur in that form in x_1, must be expressible in terms of the radicals in x_1. Therefore we must have

$$\sqrt{(l - m\sqrt{s})} = (c + d\sqrt{s}) + (g - r\sqrt{s})\sqrt{(l + m\sqrt{s})},$$

c, d, g and r being rational. Therefore

$$l - m\sqrt{s} = (c + d\sqrt{s})^2 + (g - r\sqrt{s})^2(l + m\sqrt{s}) + 2(c + d\sqrt{s})(g - r\sqrt{s})\sqrt{(l + m\sqrt{s})}.$$

Hence $(c + d\sqrt{s})(g - r\sqrt{s})$ must be zero; for, if it were not, $\sqrt{(l + m\sqrt{s})}$ would be a rational function of $\sqrt{s}$, which would make x_1 in (41) the root of a quadratic. And $g - r\sqrt{s}$ cannot be zero, for this would make

$$\sqrt{(l - m\sqrt{s})} = c + d\sqrt{s},$$

and therefore, by (41), θx_1 would be the root of a quadratic. Hence $c + d\sqrt{s}$ is zero, and therefore

$$\sqrt{(l - m\sqrt{s})} = (g - r\sqrt{s})\sqrt{(l + m\sqrt{s})}. \tag{42}$$

By comparing the first three of equations (41) with one another, it appears that the change which transforms $\sqrt{(l + m\sqrt{s})}$ into $\sqrt{(l - m\sqrt{s})}$ causes $\sqrt{(l - m\sqrt{s})}$ to become $-\sqrt{(l + m\sqrt{s})}$. Consequently, from (42),

$$-\sqrt{(l + m\sqrt{s})} = (g + r\sqrt{s})\sqrt{(l - m\sqrt{s})}. \tag{43}$$

From (42) and (43),

$$g^2 - r^2 s = -1 \;\therefore\; \sqrt{s} = \frac{\sqrt{(g^2+1)}}{r}. \tag{44}$$

By squaring both sides of (43) and equating the parts involving the radical $\sqrt{s}$,

$$2grl = m(1 + g^2 + r^2 s).$$

Therefore, by (44),

$$2grl = 2m(1 + g^2).$$

$$\therefore \quad l = \frac{m}{gr}(1 + g^2).$$

Substitute in the first of equations (41) this value of l, substituting at the same time for $\sqrt{s}$ its value in (44). Then, writing z for $1 + \left(\frac{1}{g}\right)^2$ and h for $\frac{mg}{r}$, and q for $\frac{kg}{r}$,

$$x_1 = p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})}.$$

Thus the necessity of the forms in (32) is established.

Sufficiency of the Forms.

§ 29. We now take α_1, α_2, α_4, α_3, as in (32), subject to the restriction that the quartic equation of which they are the roots must be irreducible, and we have to show that this equation is a pure uni-serial Abelian. The radical $\sqrt{(hz - h\sqrt{z})}$, which occurs in α_2, is not found in that form in α_1. But, keeping in view that $z = e^2 + 1$,

$$\sqrt{(hz - h\sqrt{z})} = \frac{\sqrt{z} - 1}{e}\sqrt{(hz + h\sqrt{z})}. \tag{45}$$

It is obvious that the expression

$$p - q\sqrt{z} + \frac{\sqrt{z} - 1}{e}\sqrt{(hz + h\sqrt{z})}$$

is a rational function of the expression

$$p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})}.$$

Therefore α_2 is a rational function of α_1; and the equation $f(x) = 0$ is a pure Abelian. That it is uni-serial may be thus shown. To pass from α_1 to α_2, we change the sign of $\sqrt{z}$, and take the resulting radical $\sqrt{(hz - h\sqrt{z})}$ with the positive sign. Let these same changes be made on α_2. The result, by (45), is

$$p + q\sqrt{z} - \frac{\sqrt{z} + 1}{e}\sqrt{(hz - h\sqrt{z})}.$$

And this again, by (45), is equivalent to

$$p + q\sqrt{z} - \frac{\sqrt{z} + 1}{e}\,\frac{\sqrt{z} - 1}{e}\sqrt{(hz + h\sqrt{z})},$$

which, because $z = e^2 + 1$, is

$$p + q\sqrt{z} - \sqrt{(hz + h\sqrt{z})}, \text{ or } \alpha_4.$$

Hence, in passing from α_1 to α_2, we pass from α_2 to α_4; and in like manner it may be shown that the same changes of the radicals carry us from α_4 to α_3 and from α_3 back to α_1; consequently the pure Abelian equation $f(x) = 0$ is uni-serial.

The Fundamental Element of the Root.

§ 30. The problem of the necessary and sufficient forms of the roots of the pure uni-serial Abelian equation of the fourth degree has been solved. We propose to find the solution by another method; and, with a view to a comparison of the result obtained above with that at which we shall arrive by the second method, we may now find expressions for R_1, the fundamental element of the root, and for the derived expressions R_0, R_2, R_3.

§ 31. By § 5 the four roots of the pure uni-serial Abelian quartic equation $f(x) = 0$ are

$$x_1 = R_0^{\frac{1}{4}} + R_1^{\frac{1}{4}} + R_2^{\frac{1}{4}} + R_3^{\frac{1}{4}},$$
$$\theta x_1 = x_2 = R_0^{\frac{1}{4}} + wR_1^{\frac{1}{4}} + w^2R_2^{\frac{1}{4}} + w^3R_3^{\frac{1}{4}},$$
$$\theta^2 x_1 = x_4 = R_0^{\frac{1}{4}} + w^2R_1^{\frac{1}{4}} + R_2^{\frac{1}{4}} + w^2R_3^{\frac{1}{4}},$$
$$\theta^3 x_1 = x_3 = R_0^{\frac{1}{4}} + w^3R_1^{\frac{1}{4}} + w^2R_2^{\frac{1}{4}} + wR_3^{\frac{1}{4}},$$

w being a primitive fourth root of unity. Therefore, because $w^2 = -1$, and $w^3 = -w$,

$$\left.\begin{aligned}
4R_0^{\frac{1}{4}} &= x_1 + x_2 + x_4 + x_3 \\
4R_1^{\frac{1}{4}} &= x_1 + w^3x_2 + w^2x_4 + wx_3 = (x_1 - x_4) - w(x_2 - x_3) \\
4R_2^{\frac{1}{4}} &= x_1 + w^2x_2 + x_4 + w^2x_3 = (x_1 + x_4) - (x_2 + x_3) \\
4R_3^{\frac{1}{4}} &= x_1 + wx_2 + w^2x_4 + w^3x_3 = (x_1 - x_4) + w(x_2 - x_3)
\end{aligned}\right\} \quad (46)$$

But, by what was proved above,

$$x_1 = p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})},$$
$$x_2 = p - q\sqrt{z} + \sqrt{(hz - h\sqrt{z})},$$
$$x_4 = p + q\sqrt{z} - \sqrt{(hz + h\sqrt{z})},$$
$$x_3 = p - q\sqrt{z} - \sqrt{(hz - h\sqrt{z})}.$$

Therefore, by (46),

$$R_0^{\frac{1}{4}} = p,$$
$$2R_1^{\frac{1}{4}} = \sqrt{(hz + h\sqrt{z})} - w\sqrt{(hz - h\sqrt{z})},$$
$$R_2^{\frac{1}{4}} = q\sqrt{z},$$
$$2R_3^{\frac{1}{4}} = \sqrt{(hz + h\sqrt{z})} + w\sqrt{(hz - h\sqrt{z})}.$$

Therefore, keeping in view that $z = e^2 + 1$, and making use of the relation $\sqrt{(hz + h\sqrt{z})}\sqrt{(hz - h\sqrt{z})} = he\sqrt{z}$,

$$\left.\begin{aligned} R_0 &= p^4 \\ 4R_1 &= h^2(e^2 + 1)(we - 1)^2 \\ R_2 &= q^4z^2 \\ 4R_3 &= h^2(e^2 + 1)(we + 1)^2 \end{aligned}\right\} \tag{47}$$

§ 32. It may not be out of place to observe that, in (47), R_1 is not presented in the form in which it is a fundamental element of the root of the pure uni-serial Abelian quartic equation $f(x) = 0$; that is to say, it is not in the form in which R_0, R_2 and R_3 can be derived from it by changing w into w^0, w^2 and w^3 respectively. In fact, by changing w in R_1, as given in (47), into w^2, we should obtain $\frac{1}{4}h^2(e^2 + 1)(e + 1)^2$; whereas, by (47), R_2 is q^4z^2 or $q^4(e^2 + 1)^2$. The form of R_1, in which it is the fundamental element of a root of a pure uni-serial Abelian quartic, will be determined afterwards.

THE PROBLEM OF THE NECESSARY AND SUFFICIENT FORMS OF THE ROOTS OF THE PURE UNI-SERIAL ABELIAN QUARTIC SOLVED FROM ANOTHER POINT OF VIEW.

The Fundamental Element of the Root.

§ 33. The necessary and sufficient forms of the roots of the pure uni-serial Abelian equation of the fourth degree may be found in another manner; namely, by making use of the principles laid down in § 5, so as to determine the fundamental element R_1 of the root. Let w be a primitive fourth root of unity. Take any rational quantities, b, c, d, m. Find the rational quantities, p, q, r, s, by means of the three equations, equivalent to four linear equations,

$$\left.\begin{aligned} p + q + r + s \quad &= d^4 \\ p - q + r - s \quad &= \frac{m^4}{(b^2 + c^2)^2} \\ (p - r) + w(q - s) &= \frac{m^2(b + cw)^2}{b^2 + c^2} \end{aligned}\right\} \tag{48}$$

Then it will be found that

$$R_1 = p + qw + rw^2 + sw^3. \tag{49}$$

b and c being rational. Therefore, from (3), taking $z = 2$,

$$R_3^{\frac{1}{2}} = (b + cw)^{-1} R_1^{\frac{1}{2}}. \tag{54}$$

Therefore, by (5), taking $e = 3$,

$$R_1^{\frac{1}{2}} = (b - cw)^{-1} R_3^{\frac{1}{2}}.$$

Therefore

$$\left.\begin{aligned} R_3^{\frac{1}{2}} &= (b^2 + c^2)^{-1} (R_1 R_3)^{\frac{1}{2}} \\ \therefore\ R_3 &= (b^2 + c^2)^{-2} (R_1 R_3) \end{aligned}\right\} \tag{55}$$

But R_1 is a rational function of w. We may put $R_1 = t + \tau w$ and $R_3 = t - \tau w$, t and τ being rational. Therefore $R_1 R_3$ is equal to the positive quantity $t^2 + \tau^2$. Therefore, from the second of equations (55), R_3 is positive.

§ 37. Because $b + cw$ and R_1 are rational functions of w, we may put

$$(b + cw)^{-2} R_1 = d + \delta w,$$

d and δ being rational. Therefore, from (54),

$$R_3 = \{(b + cw)^{-2} R_1\}^2 = d^2 - \delta^2 + 2d\delta w.$$

Since R_3 is rational, $d\delta = 0$. And δ must be zero; for, if it were not, d would be zero, and we should have $R_3 = -\delta^2$, which, because R_3 has been shown to be positive, is impossible. Therefore

$$\left.\begin{aligned} (b + cw)^{-2} R_1 &= d \\ (b - cw)^{-2} R_3 &= d \end{aligned}\right\} \tag{56}$$

Therefore also

Therefore

$$R_3 R_1^{-3} = \{d(b + cw)^2\}^{-4} \{d(b^2 + c^2)\}^2.$$

From (3), $R_3 R_1^{-3}$ is the fourth power of a rational function of w. Therefore $\{d(b^2 + c^2)\}^2$ is the fourth power of a rational function of w. Therefore

$$\pm\, d(b^2 + c^2) = (g + kw)^2 = g^2 - k^2 + 2gkw,$$

g and k being rational, the double sign on the extreme left of the equation indicating that it is not yet determined which of the two signs is to be taken. Hence $gk = 0$. Therefore $\pm\, d(b^2 + c^2)$ is equal either to g^2 or to $-k^2$. That is, $d(b^2 + c^2)$ is the square of a rational quantity, with the positive or negative sign. Hence we may put

$$d(b^2 + c^2) = m^2 w^{2s},$$

m being rational and w^{2s} having one of the two values $1, -1$. Substituting for d in (56) its value now obtained,

$$R_1 = \frac{m^2 w^{2s} (b + cw)^2}{b^2 + c^2},$$

and

$$R_3 = \frac{m^2 w^{2s} (b - cw)^2}{b^2 + c^2}.$$

But w^{2z} is either 1 or -1. In the former case,

$$R_1 = \frac{m^2 (b + cw)^2}{(b^2 + c^2)}. \tag{57}$$

In the latter case, $w^{z} = w^3$. Then

$$R_1 = \frac{m^2 (bw - c)^2}{b^2 + c^2},$$

an expression essentially of the same character as (57). Therefore (57) is the universal form of R_1. From (57),

$$R_3 = \frac{m^2 (b - cw)^2}{b^2 + c^2}.$$

Therefore $R_1 R_3 = m^4$. Hence, from (55),

$$R_2 = \frac{m^4}{(b^2 + c^2)^2}. \tag{58}$$

Let R_1, when so expressed that it is the fundamental element of the root of a pure uni-serial Abelian quartic, be

$$R_1 = p' + q'w + r'w^2 + s'w^3 = (p' - r') + w(q' - s'),$$

p', q', r' and s' being rational. Then

$$R_2 = p' + q'w^2 + r' + s'w^2 = (p' + r') - (q' + s').$$

Therefore, by (57) and (58),

$$\left.\begin{aligned} (p' + r') - (q' + s') &= \frac{m^4}{(b^2 + c^2)^2} \\ \text{and} \qquad (p' - r') + w(q' - s') &= \frac{m^2 (b + cw)^2}{b^2 + c^2} \end{aligned}\right\} \tag{59}$$

And, by §5, $R_0^{\frac{1}{4}}$ is rational. Therefore, d being some rational quantity,

$$p' + q' + r' + s' = d^4. \tag{60}$$

The equations (59) and (60) for the determination of p', q', r', s' are the same as the equations (48) for the determination of p, q, r, s. Therefore

$$p' = p, \quad q' = q, \quad r' = r, \quad s' = s.$$

Hence

$$R_1 = p + qw + rw^2 + sw^3,$$

which is the form of the fundamental element in (49). And, by §34, in constructing the root x_1 from its fundamental element, having assigned a definite character to $R_1^{\frac{1}{4}}$, we then, knowing that $R_1 R_3$ is equal to m^4, selected the value of $R_3^{\frac{1}{4}}$ so as to make $R_1^{\frac{1}{4}} R_3^{\frac{1}{4}}$ equal to m. Hence the necessity of the form of R_1 in (49) and of the relation between the roots $R_1^{\frac{1}{4}}$ and $R_3^{\frac{1}{4}}$ indicated in (51) is made

good. At the same time, because $R_1^{\frac{1}{4}}R_3^{\frac{1}{4}} = m$, $R_1^{\frac{1}{4}}R_3^{\frac{1}{4}} = m^2$; therefore, by the first of equations (55), $R_2^{\frac{1}{4}}$ is positive.

Sufficiency of the Forms.

§ 38. To prove that the above forms are sufficient, we have to show that the conditions specified in § 10 are satisfied, it being assumed that the equation of the fourth degree, of which the root is given in (53), is irreducible. *The first condition* is that $R_0^{\frac{1}{n}}$ must be rational. This is satisfied by the first of equations (48). *The next condition* is that an equation of the type (3) subsists for every integral value of z. It will be enough to consider two values of z, namely, 2 and 3. Because

$$R_1 = p + qw + rw^2 + sw^3 = (p - r) + w(q - s)$$

and

$$R_2 = p + qw^2 + r \quad + sw^2 = (p + r) - \quad (q + s),$$

we have, from the last two of equations (48),

$$R_2 R_1^{-2} = \frac{m^4}{(b^2+c^2)^2} \times \frac{m^{-4}}{(b^2+c^2)^{-2}}(b + cw)^{-4} = (b + cw)^{-4}.$$

Hence an equation of the type (3) subsists when $z = 2$. Again,

$$R_3 = p + qw^3 + rw^2 + sw = (p - r) - w(q - s).$$

But $$(p - r) + w(q - s) = \frac{m^2(b + cw)^2}{b^2 + c^2};$$

therefore $$(p - r) - w(q - s) = \frac{m^2(b - cw)^2}{b^2 + c^2}.$$

Therefore $$R_3 R_1^{-3} = m^{-4}(b + cw)^4(b - cw)^{-4}.$$

Hence an equation of the type (3) subsists when $z = 3$. Consequently an equation of the type (3) subsists for every integral value of z. *The third condition* is that equation (5) must subsist along with (3) for every value of e prime to 4. As we may leave out of view values of e greater than 4, we have only to consider the case in which $e = 3$. Also it will be enough to consider the cases in which z is equal to one of the numbers 0, 2, 3. Let $z = 0$. Then equation (3) is

$$R_0^{\frac{1}{4}} = \{F(w)\}R_1^{\frac{0}{4}} = F(w).$$

But $R_0^{\frac{1}{4}}$ is rational. Hence, changing w into w^3,

$$R_0^{\frac{1}{4}} = F(w^3).$$

Also $R_{ez}^{\frac{1}{4}} = R_0^{\frac{1}{4}}$. Therefore

$$R_{ez}^{\frac{1}{4}} = F(w^3) = \{F(w^3)\}R_3^{\frac{0}{4}}.$$

This is equation (5); so that, when $z=0$, equation (5) subsists along with (3). Next, let $z=2$. Then equation (3) is

$$R_2^{\frac{1}{4}}=\{F(w)\}R_1^{\frac{2}{4}}. \tag{61}$$

$$\therefore\ R_2=\{F(w)\}^4R_1^2.$$

Therefore, changing w into w^3,

$$R_2=\{F(w^3)\}^4R_3^2.$$

Therefore
$$R_2^{\frac{1}{4}}=w'\{F(w^3)\}R_3^{\frac{2}{4}}, \tag{62}$$

w' being an n^{th} root of unity. From (61) and (62),

$$R_2^{\frac{1}{2}}=w'\{F(w)\}\{F(w^3)\}(R_1R_3)^{\frac{1}{4}}.$$

Let $F(w)=g+hw$, g and h being rational. Therefore $F(w^3)=g-hw$. Therefore $\{F(w)\}\{F(w^3)\}$ is equal to the positive quantity g^2+h^2. Also, from the manner in which the root x_1 was constructed in § 34 from its fundamental element, $R_1^{\frac{1}{4}}R_3^{\frac{1}{4}}=m$. Therefore $(R_1R_3)^{\frac{1}{4}}$ is positive. Also, in constructing the root, $R_2^{\frac{1}{2}}$ was taken positive. Therefore w' is positive; that is, $w'=1$. Therefore, from (62),

$$R_2^{\frac{1}{4}}=\{F(w^3)\}R_3^{\frac{2}{4}}. \tag{63}$$

But, equation (61) being (3), (63) is (5); so that, when $z=2$, equation (5) subsists along with (3) Finally, let $z=3$. Then equation (3) is

$$R_3^{\frac{1}{4}}=q_1R_1^{\frac{3}{4}}, \tag{64}$$

q_1 being a rational function of w. Therefore

$$R_3=q_1^3R_1^3.$$

Therefore, changing w into w^3, and denoting by q_3 what q_1 becomes when w is changed into w^3,

$$R_1=q_3^4R_3^3.$$

Therefore
$$R_1^{\frac{1}{4}}=w'q_3R_3^{\frac{3}{4}}, \tag{65}$$

w' being one of the fourth roots of unity. From (64) and (65),

$$(R_1R_3)^{\frac{1}{2}}(q_1q_3)\,w'=1.$$

But in the same way in which the product of $F(w)$ and $F(w^3)$ was shown to be positive, q_1q_3 can be shown to be positive. Also $(R_1R_3)^{\frac{1}{4}}=m$. Therefore $(R_1R_3)^{\frac{1}{2}}=m^2$. Hence w' must be positive. Therefore $w'=1$, and (65) becomes

$$R_1^{\frac{1}{4}}=q_3R_3^{\frac{3}{4}}. \tag{66}$$

Equation (64) being (3), equation (66) is (5). Hence, whether z be zero, or 2 or 3, equation (5) subsists along with (3). Thus all the conditions specified in § 12 are satisfied, and hence, by the Criterion in § 10, x_1 is the root of a pure uni-serial Abelian quartic.

Identity of the Results Obtained by the Two Methods.

§ 39. It may be well to show that the results obtained by the two methods that have been employed for finding the necessary and sufficient forms of the roots of the pure uni-serial Abelian equation of the fourth degree are identical. In (47) we have expressions for R_1, R_2 and R_3 as determined by the first method. What we need to make out is that these are substantially the same as the expressions for R_1, R_2 and R_3 obtained by the second method. By (48),

$$R_1 = \frac{m^2(b+cw)^2}{b^2+c^2}.$$

Write $\frac{h}{2}$ for $\frac{mb^2}{b^2+c^2}$ and $-e$ for $\frac{c}{b}$. Then

$$\frac{m^2}{b^2+c^2} = \frac{h^2(b^2+c^2)}{4b^4}.$$

Also $\frac{b^2+c}{b^2} = 1 - e^2$ and $\frac{2bc}{b^2} = -2e$. Therefore

$$b^2 - c^2 + 2bcw = b^2(1 - e^2 - 2ew);$$

or

$$(b+cw)^2 = b^2(1-ew)^2.$$

Therefore $$\frac{m^2(b+cw)^2}{b^2+c^2} = \frac{h^2(b^2+c^2)}{4b^2}(1-ew)^2 = \frac{h^2}{4}(1+e^2)(1-ew)^2.$$

The expression on the extreme left of this result is the value of R_1 obtained by the second method, while that on the extreme right is the value of R_1 obtained by the first method. The value of R_3 by either method is what R_1 becomes by changing w into w^3 or $-w$; so that, when the identity of the expressions obtained for R_1 by the two methods has been established, the identity of the expressions for R_3 follows. Finally, by the second of equations (48),

$$R_2 = \frac{m^4}{(b^2+c^2)^2}.$$

The above values of h and e make this

$$R_2 = \frac{h^4}{16b^4}(1+e^2)^2.$$

Put z for $1+e^2$, and q for $\frac{h}{2b}$. Then

$$R_2 = q^4z^2,$$

which is the expression for R_2 in (47).

The Pure Uni-Serial Abelian of a Degree which is the Continued Product of a Number of Distinct Prime Numbers.

Fundamental Element of the Root.

§40. Let n be the continued product of the distinct prime numbers

$$s,\ t,\ \ldots\ldots,\ d,\ b. \tag{67}$$

Take w a primitive n^{th} root of unity. Then, if

$$\sigma,\ \tau,\ \ldots\ldots,\ \delta,\ \beta \tag{68}$$

be such that $n = s\sigma = t\tau = \ldots\ldots = b\beta$, w^σ is a primitive s^{th} root of unity, w^τ a primitive t^{th} root of unity, and so on. Let

$$\left.\begin{array}{l} w^\sigma,\ w^{\sigma\lambda},\ w^{\sigma\lambda^2},\ \ldots\ldots,\ w^{\sigma\lambda^{s-2}} \\ w^\tau,\ w^{\tau h},\ w^{\tau h^2},\ \ldots\ldots,\ w^{\tau h^{t-2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w^\delta,\ w^{\delta l},\ w^{\delta l^2},\ \ldots\ldots,\ w^{\delta l^{d-2}} \\ w^\beta,\ w^{\beta k},\ w^{\beta k^2},\ \ldots\ldots,\ w^{\beta k^{b-2}} \end{array}\right\} \tag{69}$$

be cycles containing respectively all the primitive s^{th} roots of unity, all the primitive t^{th} roots of unity, and so on. Should the numbers forming the series (67) be all odd, each of the cycles (69) consists of more terms than one. Should the prime number 2 be a term in (67), say b, the last of the cycles (69) would be reduced to the single term w^β, which it will be convenient to regard as a cycle though it consists of only one term.* In this case $k = 1$. It may be assumed that λ is less than s, h less than t, and so on as regards all the numbers s, t, etc., in (67) which are odd primes. The numbers λ, h, etc., are prime roots of s, t, etc., respectively. Take P_1 a rational function of w, and, z being any integer, let P_z be what P_1 becomes when z is changed into w^z. Put

$$\left.\begin{array}{l} \phi_\sigma = P_\sigma^{\lambda^{s-2}} P_{\sigma\lambda}^{\lambda^{s-3}} P_{\sigma\lambda^2}^{\lambda^{s-4}} \ldots\ldots P_{\sigma\lambda^{s-2}} \\ \psi_\tau = P_\tau^{h^{t-2}} P_{\tau h}^{h^{t-3}} P_{\tau h^2}^{h^{t-4}} \ldots\ldots P_{\tau h^{t-2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_\delta = P_\delta^{l^{d-2}} P_{\delta l}^{l^{d-3}} P_{\delta l^2}^{\lambda^{d-4}} \ldots\ldots P_{\delta l^{d-2}} \\ F_\beta = P_\beta^{k^{b-2}} P_{\beta k}^{k^{b-3}} P_{\beta k^2}^{k^{b-4}} \ldots\ldots P_{\beta k^{b-2}} \end{array}\right\} \tag{70}$$

In the case when one of the numbers in (67), say b, is 2, the last of equations (70) is reduced to

$$F_\beta = P_\beta. \tag{71}$$

Then, if R_1 be the fundamental element of the root of a pure uni-serial Abelian equation $f(x) = 0$ of the n^{th} degree, it will be found that

$$R_1 = A_1^n\,(\phi_\sigma^\sigma \psi_\tau^\tau \ldots\ldots X_\delta^\delta F_\beta^\beta), \tag{72}$$

A_1 being a rational function of w.

The Root Constructed from its Fundamental Element.

§41. From R_1, as expressed in (72), derive R_0, R_2, etc., by changing w into w^0, w^2, etc. By §5, the root of the equation $f(x) = 0$ is

$$R_0^{\frac{1}{n}} + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \dots + R_{n-1}^{\frac{1}{n}}. \tag{73}$$

To construct the root, we have to determine the particular n^{th} roots of R_0, R_1, etc., that are to be taken together in (73). When w is changed into w^z, let A_1, ϕ_1, ψ_1, etc., become A_z, ϕ_z, ψ_z, etc., respectively. Then

$$\left.\begin{aligned} R_z &= A_z^n(\phi_{z\sigma}^{\sigma}\psi_{z\tau}^{\tau} \dots X_{z\delta}^{\delta}F_{z\beta}^{\beta}) \\ \text{therefore} \qquad R_z^{\frac{1}{n}} &= w' A_z(\phi_{z\sigma}^{\sigma}\psi_{z\tau}^{\tau} \dots X_{z\delta}^{\delta}F_{z\beta}^{\beta})^{\frac{1}{n}} \end{aligned}\right\} \tag{74}$$

w' being an n^{th} root of unity. Let the integers not greater than n that measure n, unity not included, be

$$n,\ y,\ \text{etc.} \tag{75}$$

For instance, if $n = 3 \times 5 \times 7 = 105$, the series (75) is

$$105,\ 35,\ 21,\ 15,\ 7,\ 5,\ 3.$$

The n^{th} roots of unity distinct from unity are the primitive n^{th} roots of unity, the primitive y^{th} roots of unity, and so on. For instance, the series of the 105^{th} roots of unity distinct from unity, containing 104 terms, is made up of the 48 primitive 105^{th} roots of unity, the 24 primitive 35^{th} roots of unity, the 12 primitive 21^{st} roots of unity, the 8 primitive 15^{th} roots of unity, the 6 primitive 7^{th} roots of unity, the 4 primitive 5^{th} roots of unity, and the 2 primitive 3^{d} roots of unity. The general primitive n^{th} root of unity being w^e, give w' in the second of equations (74) the value unity for every value of z included under e. Then

$$R_e^{\frac{1}{n}} = A_e(\phi_{e\sigma}^{\sigma}\psi_{e\tau}^{\tau} \dots X_{e\delta}^{\delta}F_{e\beta}^{\beta})^{\frac{1}{n}}. \tag{76}$$

Taking any other term than n, say y, in the series (75), since y is a factor of n, let $yv = n$. Then w^v is a primitive y^{th} root of unity. Hence, since w^e is the general primitive n^{th} root of unity, all the primitive y^{th} roots of unity are included in w^{ev}. If w', in the second of equations (74), be w^a when $z = v$, let it have the value w^{ea} when $z = ev$. Then

$$R_{ev}^{\frac{1}{n}} = w^{ea}A_{ev}(\phi_{ev\sigma}^{\sigma}\psi_{ev\tau}^{\tau} \dots X_{ev\delta}^{\delta}F_{ev\beta}^{\beta})^{\frac{1}{n}}. \tag{77}$$

Form equations similar to (77) for the remaining terms in (75). In this way, because the series of the n^{th} roots of unity distinct from unity is made up of the primitive n^{th} roots of unity, the primitive y^{th} roots of unity, and so forth, all the terms $1, 2, \dots, n-1$ are found in the groups of numbers represented

by the subscripts e, ev, etc., with multiples of n rejected. Consequently, in determining $R_e^{\frac{1}{n}}$, $R_{ev}^{\frac{1}{n}}$, etc., as in (76), (77), etc., we have determined all the terms

$$R_1^{\frac{1}{n}},\ R_2^{\frac{1}{n}},\ \ldots\ldots,\ R_{n-1}^{\frac{1}{n}}. \tag{78}$$

Substitute, then, in (73) the rational value which $R_0^{\frac{1}{n}}$ can be shown, as in § 8, to possess, and the values of the terms in (78) as these are determined in (76), (77), etc., and the root is constructed; in other words, the expression (73) shall be the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it is the root, is irreducible.

Necessity of the Above Forms.

§ 42. Here we assume that the root of a pure uni-serial Abelian equation $f(x) = 0$ of the n^{th} degree is expressible as in (73), and we have to prove that its fundamental element R_1 has the form (72), and that the terms in (78) are to be taken as in (76), (77), etc., while $R_0^{\frac{1}{n}}$ receives its rational value.

§ 43. By (3), z being any integer,

$$R_{z\sigma}^{\frac{1}{n}} = \{F(w)\}\, R_1^{\frac{z\sigma}{n}},$$

$F(w)$ being a rational function of w. And equation (5) subsists along with (3); that is, w^e being the general primitive n^{th} root of unity,

$$R_{ez\sigma}^{\frac{1}{n}} = \{F(w^e)\}\, R_e^{\frac{z\sigma}{n}}.$$

Taking $z = 1$,

$$R_{e\sigma}^{\frac{\lambda^{s-2}}{n}} = B_e R_e^{\frac{\sigma\lambda^{s-2}}{n}},$$

B_e being a rational function of w^e. In like manner, taking $z = \lambda$,

$$R_{e\sigma\lambda}^{\frac{\lambda^{s-2}}{n}} = C_e R_e^{\frac{\sigma\lambda^{s-2}}{n}},$$

C_e being a rational function of w^e. In this way it can be shown that each of the terms in the series

$$R_{e\sigma}^{\frac{\lambda^{s-2}}{n}},\ R_{e\sigma\lambda}^{\frac{\lambda^{s-3}}{n}},\ R_{e\sigma\lambda^2}^{\frac{\lambda^{s-4}}{n}},\ \ldots\ldots,\ R_{e\sigma\lambda^{s-2}}^{\frac{1}{n}},$$

is the product of $R_e^{\frac{\sigma\lambda^{s-2}}{n}}$ by a rational function of w^e. Therefore

$$(R_{e\sigma}^{\lambda^{s-2}} R_{e\sigma\lambda}^{\lambda^{s-3}} R_{e\sigma\lambda^2}^{\lambda^{s-4}} \ldots\ldots R_{e\sigma\lambda^{s-2}})^{\frac{\sigma}{n}} = F_e R_e^{\frac{d}{n}} \tag{79}$$

where F_e is a rational function of w^e, and

$$d = \sigma^2 (s-1)\, \lambda^{s-2}. \tag{80}$$

So also

$$\left.\begin{array}{l} (R_{e\tau}^{h^{t-2}} R_{e\tau h}^{h^{t-3}} \ldots\ldots R_{e\tau h^{t-2}})^{\frac{\tau}{n}} = G_e R_e^{\frac{\delta}{n}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (R_{e\beta}^{k^{b-2}} R_{e\beta k}^{k^{b-3}} \ldots\ldots R_{e\beta k^{b-2}})^{\frac{\beta}{n}} = H_e R_e^{\frac{D}{n}} \end{array}\right\} \tag{81}$$

where G_e, H_e, etc., are rational functions of w^e, and

$$\left.\begin{array}{l} \delta = \tau^2 (t-1) h^{t-2} \\ \dots\dots\dots\dots \\ D = \beta^2 (b-1) k^{b-2} \end{array}\right\} \tag{82}$$

From (79) and (81),

$$(R_{e\sigma}^{\lambda^{s-2}} \dots .)^{\frac{\sigma}{n}} (R_{e\tau}^{h^{t-2}} \dots .)^{\frac{\tau}{n}} \dots . (R_{e\beta}^{k^{b-2}} \dots .)^{\frac{\beta}{n}} = Q_e R_e^{\frac{\Delta}{n}}, \tag{83}$$

where Q_e is a rational function of w^e, and Δ is the sum of the terms $d, \delta, \dots, D$; that is, by (80) and (82),

$$\Delta = \sigma^2 (s-1) \lambda^{s-2} + \tau^2 (t-1) h^{t-2} + \dots . + \beta^2 (b-1) k^{b-2}. \tag{84}$$

Because $b\beta = n = s\sigma$, and the prime numbers b and s are factors of n distinct from one another, b is a factor of σ. Hence b is a factor of the first of the separate members of the expression for Δ in (84). In like manner b is a factor of all the separate members of the expression for Δ except the last. And it is not a factor of the last. For, assuming the prime factors of n in (67) to be all odd, since the last line in (69) is a cycle of primitive b^{th} roots of unity, k is prime to b. And $b-1$ is necessarily prime to b. And β is prime to b, because β is the continued product of those prime factors of n which are distinct from b. Hence $\beta^2 (b-1) k^{b-2}$ is prime to b. The conclusion still holds if b is not odd, but equal to 2. For, in that case, $k=1$ and $b-1=1$; so that

$$\beta^2 (b-1) k^{b-2} = \beta^2.$$

Now, β^2 is odd, because β is the continued product of the odd factors of n. Hence β^2 is prime to b or 2. Whether, therefore, the terms in (67) are all odd or not, every one of the separate members of the expression for Δ in (84) except the last is divisible by b, but the last is not divisible by b. Hence Δ is prime to b. In like manner Δ is prime to each of the factors of n. Therefore it is prime to n. Therefore there are whole numbers m and r such that

$$m\Delta = rn + 1.$$

Therefore, from (83),

$$(R_{e\sigma}^{\lambda^{s-2}} \dots .)^{\frac{m\sigma}{n}} (R_{e\tau}^{h^{t-2}} \dots .)^{\frac{m\tau}{n}} \dots . (R_{e\beta}^{k^{b-2}} \dots .)^{\frac{m\beta}{n}} = (Q_e^m R_e^r) R_e^{\frac{1}{n}}.$$

For any integral value of z, let $(R_z^m)^{\frac{1}{n}}$ be written $P_z^{\frac{1}{n}}$. Then, putting A_e^{-1} for $Q_e^m R_e^r$, $R_e^{\frac{1}{n}} A_e^{-1}$ is the continued product of the expressions

$$\begin{array}{l} (P_{e\sigma}^{\lambda^{s-2}} P_{e\sigma\lambda}^{\lambda^{s-3}} \dots . P_{e\sigma\lambda^{s-2}})^{\frac{\sigma}{n}}, \\ (P_{e\tau}^{h^{t-2}} P_{e\tau h}^{h^{t-3}} \dots . P_{e\tau h^{t-2}})^{\frac{\tau}{n}}, \\ \dots\dots\dots\dots\dots \\ (P_{e\delta}^{l^{d-2}} P_{e\delta l}^{l^{d-3}} \dots . P_{e\delta l^{d-2}})^{\frac{\delta}{n}}, \\ (P_{e\beta}^{k^{b-2}} P_{e\beta k}^{k^{b-3}} \dots . P_{e\beta k^{b-2}})^{\frac{\beta}{n}}; \end{array}$$

where k_1 is a rational function of w, and k_c is what k_1 becomes by changing w into w^c. By putting $e = 1$ in (85),

$$R_1^{\frac{1}{n}} = A_1 (\phi_\sigma^\sigma \psi_\tau^\tau \ldots . X_\delta^\delta F_\beta^\beta)^{\frac{1}{n}}.$$

Taking this in connection with the second of equations (87),

$$(R_v R_1^{-v})^{\frac{1}{n}} = w^a (A_v A_1^{-v})\, Q\, (F_\beta^{-v\beta} X_\delta^{-v\delta} \ldots .)^{\frac{1}{n}} \{(\phi_{v\sigma}^\sigma \phi_\sigma^{-v\sigma})(\psi_{v\tau}^\tau \psi_\tau^{-v\tau}) \ldots . \}^{\frac{1}{n}}. \quad (89)$$

In like manner, by putting c for e in (85), and taking the result in connection with the first of equations (87),

$$(R_{cv} R_c^{-v})^{\frac{1}{n}} = w^r (A_{cv} A_c^{-v})\, Q\, (F_{c\beta}^{-v\beta} \ldots .)^{\frac{1}{n}} \{(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})(\psi_{cv\tau}^\tau \psi_{c\tau}^{-v\tau}) \ldots . \}^{\frac{1}{n}}. \quad (90)$$

From (89) compared with the first of equations (88), and from (90) compared with the second of equations (88),

$$\left.\begin{aligned} k_1 &= w^a (A_v A_1^{-v})\, Q\, (F_\beta^{-v\beta} \ldots .)^{\frac{1}{n}} \{(\phi_{v\sigma}^\sigma \phi_\sigma^{-v\sigma})(\psi_{v\tau}^\tau \psi_\tau^{-v\tau}) \ldots . \}^{\frac{1}{n}} \\ \text{and} \quad k_c &= w^r (A_{cv} A_c^{-v})\, Q\, (F_{c\beta}^{-v\beta} \ldots .)^{\frac{1}{n}} \{(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})(\psi_{cv\tau}^\tau \psi_{c\tau}^{-v\tau}) \ldots . \}^{\frac{1}{n}} \end{aligned}\right\} \quad (91)$$

By § 9, because ϕ_σ is of the same structure as the expression (8),

$$(\phi_{v\sigma} \phi_\sigma^{-v})^{\frac{1}{s}} = q_\sigma,$$

q_σ being a rational function of the primitive s^{th} root of unity w^σ. And, since it appeared from the reasoning in § 9 that the nature of the function does not depend on the particular primitive s^{th} root of unity denoted by w^σ, we have at the same time

$$(\phi_{cv\sigma} \phi_{c\sigma}^{-v})^{\frac{1}{s}} = q_{c\sigma},$$

$q_{c\sigma}$ being what q_σ becomes when w is changed into w^c. Therefore, because $s\sigma = n$,

$$(\phi_{v\sigma}^\sigma \phi_\sigma^{-v\sigma})^{\frac{1}{n}} = q_\sigma$$

and

$$(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})^{\frac{1}{n}} = q_{c\sigma}.$$

Similarly,

$$(\psi_{v\tau}^\tau \psi_\tau^{-v\tau})^{\frac{1}{n}} = q'_\tau$$

and

$$(\psi_{cv\tau}^\tau \psi_{c\tau}^{-v\tau})^{\frac{1}{n}} = q'_{c\tau},$$

where q'_τ is a rational function of w^τ, and $q'_{c\tau}$ is what q'_τ becomes when w is changed into w^c. Therefore, from (91),

$$\left.\begin{aligned} k_1 &= w^a (A_v A_1^{-v})\, Q\, (F_\beta^{-v\beta} \ldots .)^{\frac{1}{n}} (q_\sigma q'_\tau \ldots .) \\ \text{and} \quad k_c &= w^r (A_{cv} A_c^{-v})\, Q\, (F_{c\beta}^{-v\beta} \ldots .)^{\frac{1}{n}} (q_{c\sigma} q'_{c\tau} \ldots .) \end{aligned}\right\} \quad (92)$$

But again, because $b\beta = n = yv$, and y is not a multiple of b, v is a multiple of b. Therefore $v\beta$ is a multiple of $b\beta$ or n. Therefore $F_{c\beta}^{\frac{-v\beta}{n}}$ is a rational

function of w^c. In like manner $X_{c\delta}^{\frac{-v\delta}{n}}$ is a rational function of w^c, and so on. Therefore the second of equations (92) may be written

$$k_c = w^r (A_{cv} A_c^{-v})\, Q M_c (q_{c\sigma} q'_{c\tau} \ldots .),$$

where M_c is a rational function of w^c. In like manner, from the first of equations (92),
$$k_1 = w^a (A_v A_1^{-v})\, Q M_1 (q_\sigma q'_\tau \ldots .),$$
M_1 being what M_c becomes in passing from w^c to w. By §4 we can change w in this last equation into w^c. This gives us

$$k_c = w^{ac} (A_{cv} A_c^{-v})\, Q M_c (q_{c\sigma} q'_{c\tau} \ldots .).$$

Comparing this with the value of k_c previously obtained, $w^r = w^{ac}$. Therefore the first of equations (87) becomes

$$R_{cv}^{\frac{1}{n}} = w^{ac} A_{cv} Q (\phi_{cv\sigma}^{\sigma} \psi_{cv\tau}^{\tau} \ldots .)^{\frac{1}{n}}.$$

Replacing Q by $(F_{cv\beta}^{\beta} X_{cv\delta}^{\delta} \ldots .)^{\frac{1}{n}}$, and putting e for c, which we are entitled to do because w^c may be any one of the roots included under w^e,

$$R_{ev}^{\frac{1}{n}} = w^{ea} A_{ev} (\phi_{ev\sigma}^{\sigma} \psi_{ev\tau}^{\tau} \ldots . F_{ev\beta}^{\beta})^{\frac{1}{n}},$$

which is the form of $R_{ev}^{\frac{1}{n}}$ in (77).

Sufficiency of the Forms.

§45. Here we assume that R_1 has the form (72), and that the terms in (78) are determined by the equations (76), (77), etc., while $R_0^{\frac{1}{n}}$ receives its rational value. We have then to prove that the expression (73) is the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it is the root, is irreducible.

§46. *In the first place,* it has been shown that there is an n^{th} root of R_0 which has a rational value; and, by hypothesis, $R_0^{\frac{1}{n}}$ has been taken with this rational value. *In the second place,* an equation of the type (3) subsists for every integral value of z. For, let z be a multiple of n. In that case it may be taken to be zero. Then

$$(R_z R_1^{-z})^{\frac{1}{n}} = R_0^{\frac{1}{n}}. \tag{93}$$

But R_0 is the n^{th} power of a rational quantity. Therefore (93) is an equation of the type (3). If z is not a multiple of n, it may be a multiple of some of the factors of n, say b, d, etc., though not of others, say s, t, etc. Because z is a multiple of b, and $b\beta = n$, $z\beta$ is a multiple of n. Therefore $F_{z\beta} = F_0$. And F_0^{β} is the n^{th} power of a rational quantity. Therefore $F_{z\beta}^{\beta}$ is the n^{th} power of a

rational quantity. In like manner $X^{\delta}_{z\delta}$ is the n^{th} power of a rational quantity, and so on. But

$$R_z = A_z^n(F^{\beta}_{z\beta}X^{\delta}_{z\delta}\ldots\ldots)(\phi^{\sigma}_{z\sigma}\psi^{\tau}_{z\tau}\ldots\ldots).$$

Since each of the quantities $F^{\beta}_{z\beta}$, $X^{\delta}_{z\delta}$, etc., is the n^{th} power of a rational quantity, let their continued product be Q^n, Q being rational. Then

$$R_z = (A_z Q)^n(\phi^{\sigma}_{z\sigma}\psi^{\tau}_{z\tau}\ldots\ldots). \tag{94}$$

Again, because $z\beta$ is a multiple of n, $F_{\beta}^{-z\beta}$ is the n^{th} power of a rational function of w. In like manner $X_{\delta}^{-z\delta}$ is the n^{th} power of a rational function of w, and so on. Let

$$(F_{\beta}^{-z\beta}X_{\delta}^{-z\delta}\ldots\ldots) = M_1^{-n},$$

M_1 being a rational function of w. Then

$$\begin{aligned} R_1^{-z} &= A_1^{-nz}(F_{\beta}^{-\beta z}\ldots\ldots)(\phi_{\sigma}^{-\sigma z}\psi_{\tau}^{-\tau z}\ldots\ldots) \\ &= (A_1^z M_1)^{-n}(\phi_{\sigma}^{-\sigma z}\psi_{\tau}^{-\tau z}\ldots\ldots). \end{aligned} \tag{95}$$

From (94) and (95),

$$R_z R_1^{-z} = (A_z A_1^{-z})^n (QM_1^{-1})^n \{(\phi^{\sigma}_{z\sigma}\phi_{\sigma}^{-z\sigma})(\psi^{\tau}_{z\tau}\psi_{\tau}^{-z\tau})\ldots\ldots\}. \tag{96}$$

From the structure of the expression ϕ_{σ}, $\phi_{z\sigma}\phi_{\sigma}^{-z}$ is, by § 9, the s^{th} power of a rational function of w^{σ}. Therefore, because $s\sigma = n$, $\phi^{\sigma}_{z\sigma}\phi_{\sigma}^{-z\sigma}$ is the n^{th} power of a rational function of w. In like manner $\psi^{\tau}_{z\tau}\psi_{\tau}^{-z\tau}$ is the n^{th} power of a rational function of w, and so on. Therefore, from (96), $R_z R_1^{-z}$ is the n^{th} power of a rational function of w. This establishes equation (3) when z is the continued product of some of the prime factors of n, but not of all. It virtually establishes equation (3) also when z is prime to n, because this case may be regarded as included in the preceding by taking the view that the factors of n which measure z have disappeared. Thus, whether z be a multiple of n or be a multiple of some factors of n, but not of others, or be prime to n, an equation of the type (3) subsists. *In the third place*, an equation of the type (5) subsists along with (3) for every value of e that makes w^e a primitive n^{th} root of unity. For, let z be prime to n. It is then included in e. Also, since z and e are both prime to n, ze is included in e; and unity is included in e. But, from the manner in which the root was constructed from its fundamental element, $R_e^{\frac{1}{n}}$ is determined as in (76). Therefore we have the four equations

$$\begin{aligned} R_1^{\frac{1}{n}} &= A_1(\phi^{\sigma}_{\sigma}\psi^{\tau}_{\tau} \quad \ldots\ldots F^{\beta}_{\beta})^{\frac{1}{n}}, \\ R_z^{\frac{1}{n}} &= A_z(\phi^{\sigma}_{z\sigma}\psi^{\tau}_{z\tau} \quad \ldots\ldots F^{\beta}_{z\beta})^{\frac{1}{n}}, \\ R_{ez}^{\frac{1}{n}} &= A_{ez}(\phi^{\sigma}_{ez\sigma}\psi^{\tau}_{ez\tau}\ldots\ldots F^{\beta}_{ez\beta})^{\frac{1}{n}}, \\ R_e^{\frac{1}{n}} &= A_e(\phi^{\sigma}_{e\sigma}\psi^{\tau}_{e\tau} \quad \ldots\ldots F^{\beta}_{e\beta})^{\frac{1}{n}}. \end{aligned}$$

$$\left.\begin{array}{ll}\text{Therefore} & (R_z R_1^{-z})^{\frac{1}{n}} = (A_z A_1^{-z})(\phi_{z\sigma}\phi_\sigma^{-z})^{\frac{\sigma}{n}} \; \ldots . \; (F_{z\beta}F_\beta^{-z})^{\frac{\beta}{n}} \\ \text{and} & (R_{ez} R_e^{-z})^{\frac{1}{n}} = (A_{ez} A_e^{-z})(\phi_{ez\sigma}\phi_{e\sigma}^{-z})^{\frac{\sigma}{n}} \; \ldots . \; (F_{ez\beta}F_{e\beta}^{-z})^{\frac{\beta}{n}}\end{array}\right\} \quad (97)$$

Because $(\phi_{z\sigma}\phi_\sigma^{-z})^{\frac{\sigma}{n}}$ and other corresponding expressions have been shown to be rational functions of the primitive n^{th} root of unity w, the two equations (97) correspond respectively to (3) and (5). If z be not prime to n, and yet not a multiple of n, it may be taken to be ev, where v is equal to $\frac{n}{y}$, y being one of the terms in the series (75) distinct from n, and w^e being the general primitive n^{th} root of unity. Then, just as we obtained the pair of equations (97) by means of (76), we can now, by means of (77), obtain the pair of equations

$$\left.\begin{array}{l}(R_{ev} R_1^{-ev})^{\frac{1}{n}} = (A_{ev} A_1^{-ev})(\phi_{ev\sigma}\phi_\sigma^{-ev})^{\frac{\sigma}{n}} \ldots . \\ (R_{cev} R_c^{-ev})^{\frac{1}{n}} = (A_{cev} A_c^{-ev})(\phi_{cev\sigma}\phi_{c\sigma}^{-ev})^{\frac{\sigma}{n}} \ldots .\end{array}\right\} \quad (98)$$

where w^c represents any one of the primitive n^{th} roots of unity. Because such expressions as $(\phi_{ev\sigma}\phi_\sigma^{-ev})^{\frac{\sigma}{n}}$ and $(\phi_{cev\sigma}\phi_{c\sigma}^{-ev})^{\frac{\sigma}{n}}$ are rational functions of w, the two equations (98) correspond respectively to (3) and (5). Finally, should z be a multiple of n, it may be taken to be zero. Then the equation corresponding to (3) is, q_1 being a rational function of w,

$$R_z^{\frac{1}{n}} = q_1 R_1^{\frac{z}{n}}; \text{ or, since } z = 0,\ R_0^{\frac{1}{n}} = q_1.$$

But $R_0^{\frac{1}{n}}$ is rational. Therefore q_1 is rational. Therefore $q_1 = q_e$; in other words, q_1 undergoes no change when w becomes w^e. Also $R_{ez}^{\frac{1}{n}} = R_0^{\frac{1}{n}} = q_e$. Therefore, since $R_e^{\frac{z}{n}} = 1$,

$$R_{ez}^{\frac{1}{n}} = q_e R_e^{\frac{z}{n}},$$

which is the equation corresponding to (5). Therefore, whatever z be, the equation (5) subsists along with (3). Hence, by the Criterion in § 10, the expression (73) is the root of a pure uni-serial Abelian equation of the n^{th} degree.

The Pure Uni-Serial Abelian of a Degree which is Four Times the Continued Product of a Number of Distinct Odd Primes.

Fundamental Element of the Root.

§ 47. Let $n = 4m$, where m is the continued product of the distinct odd prime numbers,

$$s,\ t,\ \ldots . ,\ d,\ b. \quad (99)$$

Take

$$\sigma,\ \tau,\ \ldots . ,\ \delta,\ \beta, \quad (100)$$

such that $n = s\sigma = t\tau = \ldots\ldots = b\beta$. Let w be a primitive n^{th} root of unity. Then w^m is a primitive fourth root of unity, w^σ a primitive s^{th} root of unity, and so on. Let

$$\left.\begin{array}{l} w^\sigma,\ w^{\sigma\lambda},\ w^{\sigma\lambda^2}, \ldots\ldots,\ w^{\sigma\lambda^{s-2}} \\ w^\tau,\ w^{\tau h},\ w^{\tau h^2}, \ldots\ldots,\ w^{\tau h^{t-2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w^\delta,\ w^{\delta l},\ w^{\delta l^2}, \ldots\ldots,\ w^{\delta l^{d-2}} \\ w^\beta,\ w^{\beta k},\ w^{\beta k^2}, \ldots\ldots,\ w^{\beta k^{b-2}} \end{array}\right\} \tag{101}$$

be cycles containing respectively all the primitive s^{th} roots of unity, all the primitive t^{th} roots of unity, and so on. Let P_1 be a rational function of w, and, for any integral value of z, let P_z be what P_1 becomes by changing w into w^z. We can always take P_1 such that P_m shall have the form of the fundamental element of the root of a pure uni-serial Abelian quartic; that is, P_m may receive the form of R_1 in (49) as determined by the equations (48). For, because P_1 is a rational function of w,

$$P_1 = a + a_1 w + a_2 w^2 + \ldots\ldots + a_{n-1} w^{n-1},$$

the coefficients a, a_1, etc., being rational. Therefore

$$\begin{aligned} P_m &= a + a_1 w^m + a_2 w^{2m} + \text{etc.} \\ &= (a + a_4 + \text{etc.}) + w^m (a_1 + a_5 + \text{etc.}) + w^{2m} (a_2 + \text{etc.}) + w^{3m} (a_3 + \text{etc.}). \end{aligned}$$

This may be written

$$P_m = f + f' w^m + f'' w^{2m} + f''' w^{3m}. \tag{102}$$

All that is required in order that P_1 may be a function of the kind described is that P_m in (102) be of the same character with R_1 in (49). That is, we have to make

$$f = p,\ f' = q,\ f'' = r,\ f''' = s.$$

By means of these four linear equations, the necessary relations between the quantities a, a_1, a_2, etc., can be constituted. Having thus taken P_1 subject to the condition that P_m shall have the form of the fundamental element of the root of a pure uni-serial Abelian quartic, put

$$\left.\begin{array}{l} \phi_\sigma = P_\sigma^{\lambda^{s-2}} P_{\sigma\lambda}^{\lambda^{s-3}} P_{\sigma\lambda^2}^{\lambda^{s-4}} \ldots\ldots P_{\sigma\lambda^{s-2}} \\ \psi_\tau = P_\tau^{h^{t-2}} P_{\tau h}^{h^{t-3}} P_{\tau h^2}^{h^{t-4}} \ldots\ldots P_{\tau h^{t-2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_\delta = P_\delta^{l^{d-2}} P_{\delta l}^{l^{d-3}} P_{\delta l^2}^{l^{d-4}} \ldots\ldots P_{\delta l^{d-2}} \\ F_\beta = P_\beta^{k^{b-2}} P_{\beta k}^{k^{b-3}} P_{\beta k^2}^{k^{b-4}} \ldots\ldots P_{\beta k^{b-2}} \end{array}\right\} \tag{103}$$

the primitive 2[d] root of unity. According to our usual notation, let P_z, ϕ_z, etc., be what P_1, ϕ_1, etc., become when w is changed into w^z, z being any integer. Then, from (104),

$$\left.\begin{aligned} R_z &= A_z^n (P_{zm}^m \phi_{z\sigma}^\sigma \psi_{z\tau}^\tau \ldots\ldots F_{z\beta}^\beta) \\ R_z^{\frac{1}{n}} &= w' A_z (P_{zm}^m \phi_{z\sigma}^\sigma \psi_{z\tau}^\tau \ldots\ldots F_{z\beta}^\beta)^{\frac{1}{n}} \end{aligned}\right\} \tag{108}$$

Therefore

w' being an n^{th} root of unity. The general primitive n^{th} root of unity being w^e, give w' in the second of equations (108) the value unity for every value of z included under e. Then

$$R_e^{\frac{1}{n}} = A_e (P_{em}^m \phi_{e\sigma}^\sigma \psi_{e\tau}^\tau \ldots\ldots F_{e\beta}^\beta)^{\frac{1}{n}}. \tag{109}$$

Taking any number y distinct from n in the series (107), since y is a factor of n, let $yv = n$. Then w^v is a primitive y^{th} root of unity. Hence, since w^e is the general primitive n^{th} root of unity, all the primitive y^{th} roots of unity are included in w^{ev}. If w' in the second of equations (108) be w^a when $z = v$, give w' the value w^{ea} when $z = ev$. Then

$$R_{ev}^{\frac{1}{n}} = w^{ea} A_{ev} (P_{evm}^m \phi_{ev\sigma}^\sigma \ldots\ldots F_{ev\beta}^\beta)^{\frac{1}{n}}. \tag{110}$$

The expression P_m having the form of the fundamental element of the root of a pure uni-serial Abelian quartic, it is understood that, in (110), $P_{evm}^{\frac{m}{n}}$ or $P_{evm}^{\frac{1}{4}}$ is taken with the value which it has in the root

$$P_0^{\frac{1}{4}} + P_m^{\frac{1}{4}} + P_{2m}^{\frac{1}{4}} + P_{3m}^{\frac{1}{4}}$$

of a pure uni-serial Abelian quartic; and consequently, when v is a multiple of 2, w^{ma} must have the value unity. Form equations similar to (110) for the remaining terms in (107). In this way, because the series of the n^{th} roots of unity distinct from unity is made up of the primitive n^{th} roots of unity, and the primitive y^{th} roots of unity, and so on, all the terms $1, 2, \ldots\ldots, n-1$ will be found in the groups of numbers represented by the subscripts e, ev, etc., when multiples of n are rejected. Consequently, in determining $R_e^{\frac{1}{n}}$, $R_{ev}^{\frac{1}{n}}$, etc., as in (109), (110), etc., we have determined all the terms

$$R_1^{\frac{1}{n}},\ R_2^{\frac{1}{n}},\ \ldots\ldots,\ R_{n-1}^{\frac{1}{n}}. \tag{111}$$

Substitute, then, in (105) the rational value of $R_0^{\frac{1}{n}}$, and the terms in (111) as these are determined by the equations (109), (110), etc., and the root is constructed; that is, the expression (105) is the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it is the root, is irreducible.

Necessity of the above Forms.

§ 51. Take s, any one of the odd prime numbers in the series (99). Let a_0, a_1, a_2, etc., be rational functions of w^s. Then, because $s\sigma = n$, a_0, a_1, etc., are clear of w^σ, though they may involve the primitive fourth root of unity w^m, the primitive t^{th} root of unity w^τ, and other corresponding roots exclusive of w^σ. The terms w^σ, $w^{\sigma\lambda}$, etc., in the first of the cycles (101), being all the primitive s^{th} roots of unity, I assume that if

$$a_0 + a_1 w^\sigma + a_2 w^{\sigma\lambda} + \ldots + a_{s-1} w^{\sigma\lambda^{s-2}} = 0,$$

the coefficients a_0, a_1, etc., are all equal to one another.

§ 52. The general primitive n^{th} root of unity being w^e, $s - 1$ values of e, leaving distinct residues when multiples of s are rejected, can be found of the form

$$g\sigma + 1, \tag{112}$$

g being a whole number. For, since $s\sigma = n$, the $s - 1$ terms

$$\sigma + 1,\ 2\sigma + 1, \ldots, (s-1)\sigma + 1 \tag{113}$$

are all less than n. Of these terms, not more than one can have a measure in common with n. For suppose, if possible, that two of the terms in (113), $a\sigma + 1$ and $b\sigma + 1$, have a measure in common with n. The measure which $a\sigma + 1$ has in common with n cannot be any of the measures of σ. Therefore, since $s\sigma = n$, it must be the prime number s. We may therefore put

$$a\sigma + 1 = hs.$$

In like manner,

$$b\sigma + 1 = ks,$$

h and k being whole numbers. Therefore, assuming $a - b$ to be positive,

$$(a - b)\sigma = (h - k)s.$$

But $a - b$ is less than the prime number s. It is therefore a measure of $h - k$. Therefore σ is a multiple of s; which, because σ is four times the continued product of the odd prime factors of n exclusive of s, is impossible. Hence not more than one of the $s - 1$ terms in (113) can have a measure in common with n. In other words, $s - 2$ of the terms in (113) are prime to n. Therefore $s - 1$ of the roots

$$w,\ w^{\sigma+1},\ w^{2\sigma+1}, \ldots, w^{(s-1)\sigma+1}$$

are primitive n^{th} roots of unity. This implies that there are $s - 1$ values of g

in (112), zero included, which make $w^{g\sigma+1}$ a primitive n^{th} root of unity. Let two of these values of g be g_1 and g_2. Put

$$g_1\sigma+1=q_1s+r_1$$

and

$$g_2\sigma+1=q_2s+r_2,$$

q_1 and q_2 being whole numbers, and r_1 and r_2 whole numbers less than s. Suppose, if possible, that $r_1=r_2$; then

$$(g_1-g_2)\sigma=(q_1-q_2)s,$$

which, as above, makes σ a multiple of s, and is therefore impossible. Consequently, the $s-1$ residues after multiples of s have been rejected from the $s-1$ different values of $g\sigma+1$ are all different from one another.

§53. It can now be shown that equations

$$\left.\begin{aligned}(R_{mz}R_m^{-z})^{\frac{1}{s}}&=p_m\\ \text{and}\qquad (R_{emz}R_{em}^{-z})^{\frac{1}{s}}&=p_{em}\end{aligned}\right\} \tag{114}$$

subsist for every integral value of z and every value of e that makes w^e a primitive n^{th} root of unity, p_m being a rational function of w^m, and p_{em} being what p_m becomes when w is changed into w^e. By (3) and (5), because R_1 is the fundamental element of the root of a pure uni-serial Abelian equation of the n^{th} degree,

$$(R_{ms}R_m^{-s})^{\frac{1}{n}}=k_1,$$

and

$$(R_{emz}R_{em}^{-z})^{\frac{1}{n}}=k_e,$$

k_1 being a rational function of w, and k_e being what k_1 becomes when w is changed into w^e. Therefore

$$\left.\begin{aligned}(R_{ms}R_m^{-s})^{\frac{1}{s}}&=k_1^m\\ \text{and}\qquad (R_{emz}R_{em}^{-s})^{\frac{1}{s}}&=k_e^m\end{aligned}\right\} \tag{115}$$

In the second of these equations, give e a value, say c, falling under the form (112). Then

$$(R_{cmz}R_{cm}^{-s})^{\frac{1}{s}}=k_c^m. \tag{116}$$

Since σ is a multiple of 4, we may put $c=4d+1$. Therefore $cm=dn+m$. Therefore $w^{cm}=w^m$, and $w^{cmz}=w^{ms}$. Therefore (116) may be written

$$(R_{mz}R_m^{-s})^{\frac{1}{s}}=k_c^m.$$

This, compared with the first of equations (115), gives us

$$k_c^m=k_1^m. \tag{117}$$

Since k_1^m is a rational function of a primitive n^{th} root of unity, and the first of the cycles (101) contains all the primitive s^{th} roots of unity, we may put

$$k_1^m=a_0+a_1w^{\sigma}+a_2w^{\sigma\lambda}+\ldots\ldots+a_{s-1}w^{\sigma\lambda^{s-2}}, \tag{118}$$

where the coefficients a_0, a_1, etc., are clear of w^σ; though, for anything that has yet been proved, they may involve w^m, w^τ and other corresponding roots exclusive of w^σ. In (118), by the Corollary in §4, we can change w into w^c. This causes k_1^m to become k_c^m, and w^σ to become $w^{c\sigma}$. The coefficients a_0, a_1, etc., are rational functions of w^s, and, when w is changed into w^c, w^s becomes w^{cs}; but, by (112), $cs = gn + s$; therefore $w^{cs} = w^s$. This implies that the coefficients a_0, a_1, etc., remain unaffected when w is changed into w^c. Therefore

$$k_c^m = a_0 + a_1 w^{c\sigma} + a_2 w^{c\sigma\lambda} + \text{etc.}$$

Therefore, from (117) and (118),

$$a_1 w^{c\sigma} + a_2 w^{c\sigma\lambda} + \text{etc.} = a_1 w^\sigma + a_2 w^{\sigma\lambda} + \text{etc.} \tag{119}$$

It was proved in §52 that c may have $s-1$ values, including unity, which leave distinct residues when multiples of s are rejected. Therefore one of these residues distinct from unity must be λ, which was supposed less than s, and is not unity. Giving c in (119) the value which leaves the residue λ when multiples of s are rejected, the equation (119) becomes

$$w^{\sigma\lambda}(a_1 - a_2) + w^{\sigma\lambda^2}(a_2 - a_3) + \text{etc.} = 0.$$

Here, by §51, the coefficients $a_1 - a_2$, $a_2 - a_3$, etc., must all vanish. This implies that $a_1, a_2, \ldots, a_{s-1}$ are all equal to one another. Hence

$$k_1^m = a_0 + a_1(w^\sigma + w^{\sigma\lambda} + \text{etc.}) = a_0 - a_1. \tag{120}$$

Thus k_1^m is clear of w^σ. In like manner it can be shown to be clear of all the roots

$$w^\sigma, w^\tau, \ldots, w^\delta, w^\theta;$$

it is therefore a rational function of w^m. Let it be written p_m. Then the equations (115) become

$$(R_{mz} R_m^{-z})^{\frac{1}{s}} = p_m,$$
$$(R_{emz} R_{em}^{-z})^{\frac{1}{s}} = p_{em},$$

p_{em} being what p_m becomes when w is changed into w^e. These are the equations (114).

§54. From what has been established, it follows that R_m has the form of the fundamental element of a pure uni-serial Abelian quartic. For, by §10, all that is required in order that R_m may have such a form is that the equations (114) should subsist, and that $R_0^{\frac{1}{4}}$ should have a rational value. By §5, since R_1 is the fundamental element of the root of a pure uni-serial Abelian equation of the n^{th} degree, $R_0^{\frac{1}{n}}$ has a rational value. Therefore $R_0^{\frac{1}{4}}$ has a rational value.

§ 55. In the very same way in which (83) was established, it can be proved that

$$R_{em}^{\frac{m}{n}}(R_{e\sigma}^{\lambda^{s-2}}R_{e\sigma\lambda}^{\lambda^{s-3}}\ \ldots\ R_{e\sigma\lambda^{s-2}})^{\frac{\sigma}{n}}\ \ldots\ (R_{e\beta}^{k^{b-2}}\ldots)^{\frac{\beta}{n}}=Q_eR_e^{\frac{\Delta}{n}}, \tag{121}$$

where Q_e is a rational function of w^e, and

$$\Delta=m^2+\sigma^2(s-1)\lambda^{s-2}+\tau^2(t-1)h^{t-2}+\ldots+\beta^2(b-1)k^{b-2}. \tag{122}$$

Because m is the continued product of the odd factors of n, m^2 is odd. But each of the expressions $s-1$, $t-1$, etc., is even. Therefore Δ is odd. Therefore Δ is prime to 4. Again, because m is the continued product of the odd factors of n, it is a multiple of b. And, because $s\sigma=b\beta$, σ is a multiple of b. In like manner τ is a multiple of b. In this way all the separate members of the expression for Δ in (122) except the last are multiples of b. And, by the same reasoning as was used in § 44, $\beta^2(b-1)k^{b-2}$ is not a multiple of b. Therefore Δ is prime to b. In like manner it is prime to s, t, etc. Therefore it is prime to n. Therefore there are whole numbers v and r such that

$$v\Delta=rn+1.$$

Therefore, from (121),

$$R_{em}^{\frac{mv}{n}}(R_{e\sigma}^{\lambda^{s-2}}\ \ldots)^{\frac{\sigma v}{n}}(R_{e\tau}^{h^{t-2}}\ldots)^{\frac{\tau v}{n}}\ \ldots\ (R_{e\beta}^{k^{b-2}}\ldots)^{\frac{\beta v}{n}}=(Q_e^vR_e^r)\,R_e^{\frac{1}{n}}. \tag{123}$$

For any integral value of z, let $R_z^{\frac{v}{n}}$ be written $P_z^{\frac{1}{n}}$. Then, by (103), putting A_e^{-1} for $Q_e^vR_e^r$, (123) becomes

$$R_e^{\frac{1}{n}}=A_e(P_{em}^m\phi_{e\sigma}^\sigma\psi_{e\tau}^\tau\ \ldots\ F_{e\beta}^\beta)^{\frac{1}{n}}. \tag{124}$$

Therefore

$$R_1=A_1^n(P_m^m\phi_\sigma^\sigma\psi_\tau^\tau\ \ldots\ F_\beta^\beta). \tag{125}$$

But P_m is the same as R_m^v. Therefore, by § 54, P_m is of the form of the fundamental element of the root of a pure uni-serial Abelian quartic. Therefore the expression for R_1 in (125) is identical with that in (104), and thus the form of the fundamental element in (104) is established. Also, it was necessary to take $R_0^{\frac{1}{n}}$ with its rational value, because, by § 5, $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation $f(x)=0$. And equation (124) is identical with (109), which establishes the necessity of the forms assigned to all those expressions which are contained under $R_e^{\frac{1}{n}}$. It remains to prove that the expressions contained under $R_{ev}^{\frac{1}{n}}$, $\frac{n}{v}$ or y being a term in the series (107) distinct from n, have the forms assigned to them in (110). The details to be given here are very much a repetition of what is found in § 44; but, to prevent the confusion that might arise

from explanations and references, it is thought better to present the reasoning again with some fulness.

§ 56. Since $yv = n$, and y is not equal to n, y is the continued product of some of the factors of n, but not of them all. Let s, t, etc., be the odd factors of n of which y is a multiple; and b, d, etc., the odd factors of n of which y is not a multiple. Because $yv = n = b\beta$, and b is not a factor of y, b is a factor of v. Let $v = ab$; then $v\beta = an$. Therefore $F_{ev\beta} = F_0$. In like manner $X_{ev\delta} = X_0$, and so on as regards all those terms of the type $F_{ev\beta}$ in which $\frac{n}{\beta}$ or b is an odd factor of n, but not a factor of y. Hence, putting ev for z in the second of equations (108), and separating those factors of $R_{ev}^{\frac{1}{n}}$ that are of the type $F_{ev\beta}^{\frac{1}{n}}$ from those that are not,

$$R_{ev}^{\frac{1}{n}} = w'' A_{ev} (F_0^{\beta} X_0^{\delta} \ldots .)^{\frac{1}{n}} (P_{evm}^{m} \phi_{ev\sigma}^{\sigma} \ldots .)^{\frac{1}{n}},$$

w'' being an n^{th} root of unity. We understand that $F_0^{\frac{\beta}{n}}$, $X_0^{\frac{\delta}{n}}$, etc., are taken with the rational values which it has been proved that they admit, and, as in § 44, their continued product may be called Q. Then

$$R_{ev}^{\frac{1}{n}} = w'' A_{ev} Q (P_{evm}^{m} \phi_{ev\sigma}^{\sigma} \ldots .)^{\frac{1}{n}}. \tag{126}$$

When e is taken with the particular value c, let w'' become w^r, and when e has the value unity, let w'' become w^a. Then

$$\left.\begin{aligned} R_{cv}^{\frac{1}{n}} &= w^r A_{cv} Q (P_{cvm}^{m} \phi_{cv\sigma}^{\sigma} \ldots .)^{\frac{1}{n}} \\ \text{and} \qquad R_{v}^{\frac{1}{n}} &= w^a A_v Q (P_{vm}^{m} \phi_{v\sigma}^{\sigma} \ldots .)^{\frac{1}{n}} \end{aligned}\right\} \tag{127}$$

Because R_1 is the fundamental element of the root of a pure uni-serial Abelian equation of the n^{th} degree, equations (3) and (5) subsist together; hence, because w^c is included in w^e,

$$\left.\begin{aligned} (R_v R_1^{-v})^{\frac{1}{n}} &= k_1 \\ \text{and} \qquad (R_{cv} R_c^{-v})^{\frac{1}{n}} &= k_c \end{aligned}\right\} \tag{128}$$

where k_1 is a rational function of w, and k_c is what k_1 becomes by changing w into w^c. By putting e equal to unity in (109),

$$R_1^{\frac{1}{n}} = A_1 (P_m^{m} \phi_{\sigma}^{\sigma} \ldots . F_{\beta}^{\beta})^{\frac{1}{n}}.$$

Taking this in connection with the second of equations (127),

$$(R_v R_1^{-v})^{\frac{1}{n}} = w^a (A_v A_1^{-v}) Q (F_{\beta}^{-v\beta} \ldots .)^{\frac{1}{n}} \{(P_{vm}^{m} P_m^{-vm})(\phi_{v\sigma}^{\sigma} \phi_{\sigma}^{-v\sigma}) \ldots .\}^{\frac{1}{n}}. \tag{129}$$

In like manner, by putting c for e in (109), and taking the result in connection with the first of equations (127),

$$(R_{cv}R_c^{-v})^{\frac{1}{n}} = w^r(A_{cv}A_c^{-v})Q(F_{c\beta}^{-v\beta}\ldots.)^{\frac{1}{n}}\{(P_{cvm}^m P_{cm}^{-vm})(\phi_{cv\sigma}^{\sigma}\phi_{c\sigma}^{-v\sigma})\ldots.\}^{\frac{1}{n}}. \tag{130}$$

From (129) compared with the first of equations (128), and from (130) compared with the second of equations (128),

$$\left.\begin{aligned} k_1 &= w^a(A_vA_1^{-v})Q(F_\beta^{-v\beta}\ldots.)^{\frac{1}{n}}\{(P_{vm}^m P_m^{-vm})(\phi_{v\sigma}^{\sigma}\phi_\sigma^{-v\sigma})\ldots.\}^{\frac{1}{n}} \\ \text{and}\quad k_c &= w^r(A_{cv}A_c^{-v})Q(F_{c\beta}^{-v\beta}\ldots.)^{\frac{1}{n}}\{(P_{cvm}^m P_{cm}^{-vm})(\phi_{cv\sigma}^{\sigma}\phi_{c\sigma}^{-v\sigma})\ldots.\}^{\frac{1}{n}} \end{aligned}\right\} \tag{131}$$

Exactly as in § 44, it can be shown that

$$(\phi_{cv\sigma}^{\sigma}\phi_{c\sigma}^{-v\sigma})^{\frac{1}{n}} = q_c, \tag{132}$$

q_c being a rational function of the primitive n^{th} root of unity w^c. Also, it has been proved that P_m is of the form of the fundamental element of the root of a pure uni-serial Abelian quartic. Therefore, by (3), $(P_{cvm}P_{cm}^{-v})^{\frac{1}{4}}$ is a rational function of the primitive fourth root of unity w^m. Therefore, because $n = 4m$, $(P_{cvm}^m P_{cm}^{-vm})^{\frac{1}{n}}$ is a rational function of the primitive n^{th} root of unity w^c. Put

$$(P_{cvm}^m P_{cm}^{-vm})^{\frac{1}{n}} = q'_c. \tag{133}$$

Again, exactly as in § 44,

$$F_{cv}^{-\frac{v\beta}{n}} = q''_c, \tag{134}$$

q''_c being a rational function of w^c. By (132), (133), (134), and other corresponding equations, the second of equations (131) becomes

$$k_c = w^r(A_{cv}A_c^{-v})\,Q\,(q_cq'_cq''_c\ldots.). \tag{135}$$

In like manner, from the first of equations (131),

$$k_1 = w^a(A_vA_1^{-v})\,Q\,(q_1q'_1q''_1\ldots.),$$

q_1, q'_1, etc., being what q_c, q'_c, etc., become in passing from w^c to w. It may be noted that this assumes that we are entitled to change equation (133) into

$$(P_{vm}^m P_m^{-vm})^{\frac{1}{n}} = q'_1.$$

The warrant for this lies in the fact that the roots $P_m^{\frac{m}{n}}$, $P_{2m}^{\frac{m}{n}}$, $P_{3m}^{\frac{m}{n}}$, or $P_m^{\frac{1}{4}}$, $P_{2m}^{\frac{1}{4}}$, $P_{3m}^{\frac{1}{4}}$, were taken with the values they have in the root

$$P_0^{\frac{1}{4}} + P_m^{\frac{1}{4}} + P_{2m}^{\frac{1}{4}} + P_{3m}^{\frac{1}{4}}$$

of a pure uni-serial Abelian quartic. This being so, the equation

$$(P_{vm}^m P_m^{-vm})^{\frac{1}{n}} = q'_1$$

corresponds to equation (3), while (133) corresponds to (5), and, by § 5, equations

$$R_1^{\frac{1}{n}} = A_1 (P_m^m \phi_\sigma^\sigma \quad \ldots . \, F_\beta^\beta)^{\frac{1}{n}},$$
$$R_z^{\frac{1}{n}} = A_z (P_{zm}^m \phi_{z\sigma}^\sigma \quad \ldots . \, F_{z\beta}^\beta)^{\frac{1}{n}},$$
$$R_{ez}^{\frac{1}{n}} = A_{ez} (P_{ezm}^m \phi_{ez\sigma}^\sigma \ldots . \, F_{ez\beta}^\beta)^{\frac{1}{n}}.$$

Therefore
$$\left.\begin{aligned} (R_z R_1^{-z})^{\frac{1}{n}} &= (A_z A_1^{-z}) (P_{zm} P_m^{-z})^{\frac{m}{n}} (\phi_{z\sigma} \phi_\sigma^{-z})^{\frac{\sigma}{n}} \ldots . \\ (R_{ez} R_e^{-z})^{\frac{1}{n}} &= (A_{ez} A_e^{-z})(P_{ezm} P_{em}^{-z})^{\frac{m}{n}} (\phi_{ez\sigma} \phi_{e\sigma}^{-z})^{\frac{\sigma}{n}} \ldots . \end{aligned}\right\} \tag{136}$$
and

Because $(P_{zm} P_m^{-z})^{\frac{m}{n}}$ and other such expressions have been shown to be rational functions of the primitive n^{th} root of unity, the two equations (106) correspond respectively to (3) and (5). If z be not prime to n, and yet not a multiple of n, it may be taken to be ev, where v is equal to $\frac{n}{y}$, y being one of the terms in the series (107) distinct from n, and w^e being the general primitive n^{th} root of unity. Then, just as we obtained the pair of equations (136) by means of (109), we can now, by means of (110), obtain

$$\left.\begin{aligned} (R_{ev} R_1^{-ev})^{\frac{1}{n}} &= (A_{ev} A_1^{-ev}) (P_{evm} P_m^{-ev})^{\frac{m}{n}} \ldots . \\ (R_{cev} R_c^{-ev})^{\frac{1}{n}} &= (A_{cev} A_c^{-ev})(P_{cevm} P_{cm}^{-ev})^{\frac{m}{n}} \ldots . \end{aligned}\right\} \tag{137}$$

where w^c represents any one of the primitive n^{th} roots of unity. Because $(P_{evm} P_m^{-ev})^{\frac{m}{n}}$ and other such expressions have been shown to be rational functions of the primitive n^{th} root of unity, the two equations (137) correspond respectively to (3) and (5). Finally, should z be a multiple of n, it may be taken to be zero. Then the equation corresponding to (3) is

$$R_z^{\frac{1}{n}} = q_1 R^{\frac{z}{n}},$$

q_1 being a rational function of w. Or, since $z = 0$,

$$R_0^{\frac{1}{n}} = q_1.$$

But $R_0^{\frac{1}{n}}$ is rational. Therefore q_1 is rational. Hence, if q_e be what q_1 becomes in passing from w to w^e, $q_e = q_1$. Also $R_{ez}^{\frac{1}{n}} = R_0^{\frac{1}{n}} = q_e$. Therefore, since $R_e^{\frac{z}{n}} = 1$,

$$R_{ez}^{\frac{1}{n}} = q_e R_e^{\frac{z}{n}},$$

which is the equation corresponding to (5). Therefore, whatever z be, the equation (5) subsists along with (3). Hence, by the Criterion in § 10, the expression (105) is the root of a pure uni-serial Abelian equation of the n^{th} degree.

SOLVABLE IRREDUCIBLE EQUATIONS OF PRIME DEGREES.

§58. Let $f(x) = 0$ be a solvable irreducible equation of the prime degree n. Even if it be not a pure Abelian, the necessary and sufficient forms of its roots can, by means of the problems solved above, be determined *in all cases in which n is either the continued product of a number of distinct primes or four times the continued product of a number of distinct odd primes.*

§59. It is known that the root of the equation is of the form

$$k + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \dots + R_{n-1}^{\frac{1}{n}}, \tag{138}$$

where k is rational; and

$$R_1, R_2, \dots, R_{n-1}, \tag{139}$$

are the roots of an equation of the n^{th} degree, that is, of an equation with rational coefficients. Let this equation be $\phi(x) = 0$. The root of the equation $f(x) = 0$ may also be expressed in the form

$$k + R_1^{\frac{1}{n}} + a_1 R_1^{\frac{2}{n}} + b_1 R_1^{\frac{3}{n}} + \dots + c_1 R_1^{\frac{n-1}{n}}, \tag{140}$$

where a_1, b_1, etc., are rational functions of R_1. The separate members of the expression (140) are severally equal to those of the expression (138); that is,

$$R_2^{\frac{1}{n}} = a_1 R_1^{\frac{2}{n}},\ R_3^{\frac{1}{n}} = b_1 R_1^{\frac{3}{n}},\ \dots,\ R_{n-1}^{\frac{1}{n}} = c_1 R_1^{\frac{n-1}{n}}. \tag{141}$$

Therefore $R_2 = a_1^n R_1^2$. Hence, since a_1 is a rational function of R_1, R_2 is a rational function of R_1. The expression R_1 is thus the root of a pure Abelian equation, which, moreover, is known to be capable of having its roots arranged in a single circulating series, and therefore to be what we have called a pure uni-serial Abelian. A quotation from a remarkable memoir which was presented in 1853 by Herr Leopold Kronecker to the Academy of Berlin, and of which a translation is given in Serret's Cours d'Algèbre Supérieure (Vol. II, p. 654, 3d edition), will show how the case stands. In Kronecker's memoir μ indicates the degree of the equation, and is therefore our n, while A, B, C, etc., are quantities involved rationally in the coefficients of the equation $f(x) = 0$. Having given, after Abel, what are substantially the two forms (138) and (140), Kronecker adds: "Il est bien vrai que toute fonction algébrique, satisfaisant au problème proposé, doit pouvoir se mettre sous ces deux formes; mais ces formes sont encore trop générales, c'est-à-dire qu'elles renferment des fonctions algébriques qui ne répondent pas à la question. Je les ai donc étudiées de plus près, et j'ai trouvé d'abord que parmi les fonctions renfermées dans la forme (2)" [the

same as (138)] "celles qui satisfont au problème proposé doivent avoir la propriété nonseulment que les fonctions symétriques de R_1, R_2, etc., soient rationnelles en A, B, C, etc. (ce qu'Abel a remarqué), mais aussi que les fonctions cycliques des quantités R_1, R_2, etc., prises dans un certain ordre, soient également rationnelles en A, B, C, etc.; en d'autres termes, l'équation de degré $\mu - 1$, dont R_1, R_2, etc., sont les racines, doit être une équation abélienne. J'entendrai toujours ici par équations abéliennes cette classe particulière d'équations résòluble qu'Abel a considérées dans le Memoire XI du premier volume des *Œuvres complètes*, et dont je supposerai les coefficients fonctions rationnelles de A, B, C, etc. En désignant par $x_1, x_2, \ldots, x_n$, des racines prises dans un ordre déterminé, ces équations peuvent être définies soit en disant que les fonctions cycliques des racines sont rationnelles en A, B, C, etc., soit en disant qu'on a les relations,

$$x_2 = \theta(x_1),\ x_3 = \theta(x_2),\ \ldots,\ x_n = \theta(x_{n-1}),\ x_1 = \theta x_n,$$

où $\theta(x)$ est une fonction entière de x dont les coefficients sont rationnels en A, B, C, etc." In saying that the $\mu - 1$ (or, in our notation, the $n - 1$) terms, R_1, R_2, etc., are the roots of an Abelian equation, Kronecker must be understood to assume that the equation $\phi(x) = 0$, which has the terms in (139) for its roots, is irreducible. As a matter of fact, in the most general case, which includes all the others, the equation $\phi(x) = 0$ is irreducible. But in particular cases it may be reducible, and then it is not an Abelian. In a paper by the present writer, entitled "Principles of the Solution of Equations of the Higher Degrees," which appeared in this Journal (Vol. VI, No. 1), it was proved that when the equation $\phi(x) = 0$ is reducible, it can be broken into a number of irreducible equations,

$$\psi_1(x) = 0,\ \psi_2(x) = 0,\ \ldots,\ \psi_s(x) = 0,$$

each a pure uni-serial Abelian. Hence, for a detailed discussion of the problem we have now before us, we should require to deal not only with the general case in which the equation $\phi(x) = 0$ is irreducible, but also with the several cases in which equations such as $\psi_1(x) = 0$, $\psi_2(x) = 0$, etc., can be formed. But since, as has been stated above, the particular cases are included in the general, we shall confine ourselves to the problem of the necessary and sufficient forms of the roots of the solvable irreducible equation $f(x) = 0$ of degree n, when the subordinate equation $\phi(x) = 0$ of degree $n - 1$ is irreducible, and is therefore a pure uni-serial Abelian; it being understood that $n - 1$ is either the continued product of a number of distinct primes, or four times the continued product of a number of distinct odd primes.

Form of the Root.

§ 60. The solutions of the problems investigated in the preceding part of the paper have furnished us with the necessary and sufficient form of the root of the pure uni-serial Abelian equation $\phi(x) = 0$ of degree $n - 1$. Let this be r_1. Let

$$w,\ w^{\lambda},\ w^{\lambda^2},\ \ldots\ldots,\ w^{\lambda^{n-2}} \tag{142}$$

be a cycle containing all the primitive n^{th} roots of unity. We may assume that λ is less than n. Let

$$1,\ \lambda,\ \alpha,\ \beta,\ \ldots\ldots,\ \delta,\ \varepsilon,\ \theta \tag{143}$$

be the indices of the powers of w in (143); that is, $\alpha = \lambda^2$, $\beta = \lambda^3$, and so on. The $n - 1$ roots of the equation $\phi(x) = 0$ can be arranged in a single circulating series. Let them, so arranged, be

$$r_1,\ r_\lambda,\ r_\alpha,\ \ldots\ldots,\ r_\varepsilon,\ r_\theta. \tag{144}$$

It will be found that the terms $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc., in (138), which are the same, in a certain order, as $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, $R_\alpha^{\frac{1}{n}}$, etc., with multiples of n rejected from the subscripts, are given by the equations

$$\left.\begin{aligned}
R_1^{\frac{1}{n}} &= A_1\left(r_1^{\theta} r_\lambda^{\varepsilon} r_\alpha^{\delta} \ldots\ldots r_\delta^{\alpha} r_\varepsilon^{\lambda} r_\theta\right)^{\frac{1}{n}} \\
R_\lambda^{\frac{1}{n}} &= A_\lambda\left(r_1 r_\lambda^{\theta} r_\alpha^{\varepsilon} \ldots\ldots\ r_\varepsilon^{\alpha} r_\theta^{\lambda}\right)^{\frac{1}{n}} \\
R_\alpha^{\frac{1}{n}} &= A_\alpha\left(r_1^{\lambda} r_\lambda r_\alpha^{\theta} \ldots\ldots\ r_\theta^{\alpha}\right)^{\frac{1}{n}} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
R_\theta^{\frac{1}{n}} &= A_\theta\left(r_1^{\varepsilon} r_\lambda^{\delta}\ \ldots\ldots\ r_\theta^{\theta}\right)^{\frac{1}{n}}
\end{aligned}\right\} \tag{145}$$

In (145) the subscripts of the factors of the expression for $R_1^{\frac{1}{n}} A_1^{-1}$ are the terms in (143), while the indices are the terms in (143) in reverse order. Because the series (144) circulates, R_λ is formed from R_1 by changing r_1 into r_λ, and, through the same change, R_λ becomes R_α, and so on.

Necessity of the above Forms.

§ 61. Here, assuming that the root of a solvable irreducible equation of degree n is expressible as in (138), we have to show that $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc., have the forms (145).

§ 62. In (138) $R_1^{\frac{1}{n}}$ is an n^{th} root of R_1, one of the roots of a pure uni-serial Abelian equation $\phi(x) = 0$, the series of whose roots is contained in (139). But

R_1 may be any one of the roots. This implies that if the roots, in the order in which they circulate, are

$$R_1,\ R_\lambda,\ R_\alpha,\ \ldots\ldots,\ R_\delta,\ R_\epsilon,\ R_\theta,$$

the change of $R_1^{\frac{1}{n}}$ in the system of equations (141) into $R_\lambda^{\frac{1}{n}}$ will cause $R_\lambda^{\frac{1}{n}}$ to become $R_\alpha^{\frac{1}{n}}$, and $R_\alpha^{\frac{1}{n}}$ to become $R_\beta^{\frac{1}{n}}$, and so on. In fact, by exactly the same reasoning as that used in establishing the Criterion of pure uni-serial Abelianism, it can be made to appear that the n values of the expression (138) or of (140) obtained by taking the n values of $R_1^{\frac{1}{n}}$ for a given value of R_1, and taking at the same time the appropriate values of $R_2^{\frac{1}{n}}$, $R_3^{\frac{1}{n}}$, etc., as determined by the equations (141), would not be the roots of an equation of the n^{th} degree with rational coefficients unless $R_\lambda^{\frac{1}{n}}$ could replace $R_1^{\frac{1}{n}}$ in the manner above indicated. In like manner, by changing $R_1^{\frac{1}{n}}$ in the system of equations (141) into $R_\alpha^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$ becomes $R_\beta^{\frac{1}{n}}$, and so on. The principle can be extended to all the terms in the series

$$R_1^{\frac{1}{n}},\ R_\lambda^{\frac{1}{n}},\ R_\alpha^{\frac{1}{n}},\ \ldots\ldots,\ R_\epsilon^{\frac{1}{n}},\ R_\theta^{\frac{1}{n}}. \tag{146}$$

§ 63. Let, then, the system of equations (141) be written

$$R_{e\lambda}^{\frac{1}{n}} = a'_e R_e^{\frac{\lambda}{n}},\ R_{e\alpha}^{\frac{1}{n}} = b'_e R_e^{\frac{\alpha}{n}},\ \text{etc.}, \tag{147}$$

e being a general symbol under which all the terms in the series (143) are contained, while a'_e, b'_e, etc., are rational functions of R_e. These equations give us

$$(R_e^\theta R_{e\lambda}^\epsilon R_{e\alpha}^\delta \ldots\ldots R_{e\delta}^\alpha R_{e\epsilon}^\lambda R_{e\theta})^{\frac{1}{n}} = G_e R_e^{\frac{t}{n}},$$

where G_e is a rational function of R_e, and

$$t = \theta + \epsilon\lambda + \delta\alpha + \ldots\ldots + \theta = (n-1)\,\theta = (n-1)\,\lambda^{n-2}.$$

Because λ is a prime root of n, $(n-1)\,\lambda^{n-2}$ is prime to n. Therefore t is prime to n. Therefore whole numbers h and k exist such that

$$ht = kn + 1.$$

Therefore $$(R_e^\theta R_{e\lambda}^\epsilon \ldots\ldots R_{e\theta})^{\frac{h}{n}} = (G_e^h R_e^k)\, R_e^{\frac{1}{n}}.$$

For every integral value of z, let $(R_{ez}^h)^{\frac{1}{n}}$ be written $r_{ez}^{\frac{1}{n}}$. Then, putting A_e^{-1} for $G_e^h R_e^k$,

$$R_e^{\frac{1}{n}} = A_e\,(r_e^\theta r_{e\lambda}^\epsilon r_{e\alpha}^\delta \ldots\ldots r_\delta^\alpha r_\epsilon^\lambda r_\theta)^{\frac{1}{n}}. \tag{148}$$

Because r_{es} is simply another way of writing R^h_{es}, and the terms R_1, R_λ, etc., are the roots of a pure uni-serial Abelian, it follows that r_1, r_λ, etc., have the forms of the roots of a pure uni-serial Abelian. By putting e, then, in (148) successively equal to $1, \lambda, \alpha, \ldots\ldots, \theta$, the $n-1$ terms in (146) are obtained with the forms assigned to them in (145).

Sufficiency of the Forms.

§ 64. We here assume that the terms forming the series (146) are taken as in (145), and we have to show that the expression (140) is the root of a solvable irreducible equation of the n^{th} degree; provided always that the equation of the n^{th} degree, of which it is a root, is irreducible. Because the terms forming the series (146) are taken as in (145), the system of equations (147) subsists. Therefore, by a course of reasoning precisely similar to that used in an earlier part of the paper to show that the n values of the expression (2), obtained by giving s successively the values $0, 1, 2, \ldots\ldots, n-1$, are the roots of an equation of the n^{th} degree, it can now be shown that the n values of the expression (140), obtained by taking the n values of $R_1^{\frac{1}{n}}$ for a given value of R_1, are the roots of an equation of the n^{th} degree, that is, of an equation of the n^{th} degree with rational coefficients.

Symmetric Functions of the 14^{ic}.

By William Pitt Durfee, *Hobart College.*

In the following table the functions are arranged according to the second of the two methods pointed out by me in the *American Journal of Mathematics*, Vol. V, p. 349. The symmetric functions of the 12^{ic} there published, as well as Capt. MacMahon's 13^{ic}, Vol. VI, p. 289, are arranged according to the second of the two methods. My reason for following a different method is that the order now chosen approaches more nearly to the natural or dictionary order of the partitions.

As checks or verifications of my results, I have employed symmetry, the fact that the sum of the coefficients of any column must be equal to the coefficient of the corresponding function in the value of H_{14}, and MacMahon's formula given in Proc. Lond. Math. Soc. and repeated in his introduction to the Symmetric Functions of the 13^{ic}.

	14	13.1	12.2	12.1^2	11.3	11.21	11.1^3	10.4	10.31	10.2^2	10.21^2	101^4	95	941	932
14	− 14	+ 14	+ 14	− 14	+ 14	− 28	+ 14	+ 14	− 28	− 14	+ 42	− 14	+ 14	− 28	− 28
13.1	+ 14	− 1	− 14	+ 1	− 14	+ 15	− 1	− 14	+ 15	+ 14	− 16	+ 1	− 14	+ 15	+ 28
12.2	+ 14	− 14	+ 10	+ 2	− 14	+ 4	− 2	− 14	+ 28	− 10	− 6	+ 2	− 14	+ 28	+ 4
12.1^2	− 14	+ 1	+ 2	− 1	+ 14	− 3	+ 1	+ 14	− 15	− 2	+ 4	− 1	+ 14	− 15	− 16
11.3	+ 14	− 14	− 14	+ 14	+ 19	− 5	− 3	− 14	− 5	+ 14	− 9	+ 3	− 14	+ 28	− 5
11.21	− 28	+ 15	+ 4	− 3	− 5	− 8	+ 3	+ 28	− 10	− 4	+ 11	− 3	+ 28	− 43	+ 1
11.1^3	+ 14	− 1	− 2	+ 1	− 3	+ 8	− 1	− 14	+ 4	+ 2	− 4	+ 1	− 14	+ 15	+ 5
10.4	+ 14	− 14	− 14	+ 14	− 14	+ 28	− 14	+ 26	− 12	− 6	− 2	+ 4	− 14	− 12	+ 28
10.31	− 28	+ 15	+ 28	− 15	− 5	− 10	+ 4	− 12	0	− 8	+ 15	− 4	+ 28	− 3	− 23
10.2^2	− 14	+ 14	− 10	− 2	+ 14	− 4	+ 2	− 6	− 8	0	+ 6	− 2	+ 14	− 8	− 4
10.21^2	+ 42	− 16	− 6	+ 4	− 9	+ 11	− 4	− 2	+ 15	+ 6	− 15	+ 4	− 42	+ 18	+ 15
10.1^4	− 14	+ 1	+ 2	− 1	+ 3	− 3	+ 1	+ 4	− 4	− 2	+ 4	− 1	+ 14	− 5	− 5
95	+ 14	− 14	− 14	+ 14	− 14	+ 28	− 14	− 14	+ 28	+ 14	− 42	+ 14	+ 31	− 17	− 17
941	− 28	+ 15	+ 28	− 15	+ 28	− 43	+ 15	− 12	− 3	− 8	+ 18	− 5	− 17	+ 6	− 11
932	− 28	+ 28	+ 4	− 16	− 5	+ 1	+ 5	+ 28	− 23	− 4	+ 15	− 5	− 17	− 11	− 8
931^2	+ 42	− 16	− 30	+ 16	− 9	+ 13	− 5	− 2	+ 15	+ 10	− 19	+ 5	+ 3	+ 9	+ 21
92^{2}1	+ 42	− 29	+ 6	+ 5	− 9	+ 12	− 5	− 22	+ 18	+ 4	− 17	+ 5	+ 3	+ 24	+ 12
921^3	− 56	+ 17	+ 8	− 5	+ 12	− 14	+ 5	+ 16	− 19	− 8	+ 19	− 5	+ 11	− 24	− 20
91^5	+ 14	− 1	− 2	+ 1	− 3	+ 3	− 1	− 4	+ 4	+ 2	− 4	+ 1	− 5	+ 5	+ 5
86	+ 14	− 14	− 14	+ 14	− 14	+ 28	− 14	− 14	+ 28	+ 14	− 42	+ 14	− 14	+ 28	+ 28
851	− 28	+ 15	+ 28	− 15	+ 28	− 43	+ 15	+ 28	− 43	− 28	+ 58	− 15	− 17	+ 2	− 11
842	− 28	+ 28	+ 4	− 16	+ 28	− 32	+ 16	− 12	− 16	+ 16	+ 8	− 6	+ 28	− 16	− 32
841^2	+ 42	− 16	− 30	+ 16	− 42	+ 46	− 16	− 2	+ 18	+ 10	− 22	+ 6	+ 3	+ 9	+ 27
83^2	− 14	+ 14	+ 14	− 14	− 19	+ 5	+ 3	+ 14	+ 5	− 14	+ 9	− 3	+ 14	− 28	+ 5
8321	+ 84	− 58	− 86	+ 34	+ 15	+ 17	− 12	− 44	+ 33	+ 16	− 41	+ 12	− 39	+ 57	+ 30
831^3	− 56	+ 17	+ 32	− 17	+ 12	− 16	+ 6	+ 16	− 19	− 12	+ 23	− 6	+ 11	− 24	− 26
82^3	+ 14	− 14	+ 10	+ 2	− 14	+ 4	− 2	+ 6	+ 8	0	− 6	+ 2	− 14	+ 8	+ 4
82^{2}1^2	− 84	+ 45	0	− 9	+ 18	− 23	+ 9	+ 24	− 33	− 10	+ 32	− 9	+ 39	− 42	− 27
821^4	+ 70	− 18	− 10	+ 6	− 15	+ 17	− 6	− 20	+ 23	+ 10	− 23	+ 6	− 25	+ 29	+ 25
81^6	− 14	+ 1	+ 2	− 1	+ 3	− 3	+ 1	+ 4	− 4	− 2	+ 4	− 1	+ 5	− 5	− 5
7^2	+ 7	− 7	− 7	+ 7	− 7	+ 14	− 7	− 7	+ 14	+ 7	− 21	+ 7	− 7	+ 14	+ 14
761	− 28	+ 15	+ 28	− 15	+ 28	− 43	+ 15	+ 28	− 43	− 28	+ 58	− 15	+ 28	− 43	− 56
752	− 28	+ 28	+ 4	− 16	+ 28	− 32	+ 16	+ 28	− 56	− 4	+ 48	− 16	− 17	− 11	+ 13
751^2	+ 42	− 16	− 30	+ 16	− 42	+ 46	− 16	− 42	+ 58	+ 30	− 62	+ 16	+ 3	+ 13	+ 27
743	− 28	+ 28	+ 28	− 28	− 5	− 23	+ 17	− 12	+ 17	− 8	+ 11	− 7	+ 28	− 16	− 23
7421	+ 84	− 58	− 86	+ 34	− 51	+ 83	− 34	− 4	+ 29	− 4	− 87	+ 14	− 39	+ 53	+ 42
741^3	− 56	+ 17	+ 32	− 17	+ 45	− 49	+ 17	+ 16	− 22	− 12	+ 26	− 7	+ 11	− 24	− 32
73^{2}1	+ 42	− 29	− 42	+ 29	+ 24	+ 5	− 7	− 2	− 5	+ 22	− 24	+ 7	− 42	+ 31	+ 18
732^2	+ 42	− 42	+ 6	+ 18	− 9	+ 3	− 7	− 22	+ 31	+ 4	− 21	+ 7	+ 3	+ 19	+ 12
7321^2	− 168	+ 90	+ 72	− 54	+ 3	− 41	+ 21	+ 48	− 63	− 32	+ 75	− 21	+ 78	− 84	− 66
731^4	+ 70	− 18	− 34	+ 18	− 15	+ 19	− 7	− 20	+ 23	+ 14	− 27	+ 7	− 25	+ 29	+ 31
72^{3}1	− 56	+ 43	− 16	− 7	+ 23	− 16	+ 7	+ 16	− 26	− 4	+ 23	− 7	+ 11	− 32	− 16
72^{2}1^3	+ 140	− 62	− 8	+ 14	− 30	+ 37	− 14	− 40	+ 52	+ 18	− 51	+ 14	− 50	+ 66	+ 47
6^{2}2	− 14	+ 14	+ 2	− 8	+ 14	− 16	+ 8	+ 14	− 28	− 2	+ 24	− 8	+ 14	− 28	− 16
6^{2}1^2	+ 21	− 8	− 15	+ 8	− 21	+ 23	− 8	− 21	+ 29	+ 15	− 31	+ 8	− 21	+ 29	+ 36

	931^2	92^21	921^3	91^5	86	851	842	841^2	83^2	8321	831^3	82^3	82^21^2	821^4	81^6
14	+42	+42	−56	+14	+14	−28	−28	+42	−14	+84	−56	+14	−84	+70	−14
13.1	−16	−29	+17	−1	−14	+15	+28	−16	+14	−58	+17	−14	+45	−18	+1
12.2	−30	+6	+8	−2	−14	+28	+4	−30	+14	−36	+32	+10	0	−10	+2
12.1^2	+16	+5	−5	+1	+14	−15	−16	+16	−14	+34	−17	+2	−9	+6	−1
11.3	−9	−9	+12	−3	−14	+28	+28	−42	−19	+15	+12	−14	+18	−15	+3
11.21	+13	+12	−14	+3	+28	−43	−32	+46	+5	+17	−16	+4	−23	+17	−3
11.1^3	−5	−5	+5	−1	−14	+15	+16	−16	+3	−12	+6	−2	+9	−6	+1
10.4	−2	−22	+16	−4	−14	+28	−12	−2	+14	−44	+16	+6	+24	−20	+4
10.31	+15	+18	−19	+4	+28	−48	−16	+18	+5	+33	−19	+8	−33	+28	−4
10.2^2	+10	+4	−8	+2	+14	−28	+16	+10	−14	+16	−12	0	−10	+10	−2
10.21^2	−19	−17	+19	−4	−42	+58	+8	−22	+9	−41	+23	−6	+32	−23	+4
10.1^4	+5	+5	−5	+1	+14	−15	−6	+6	−3	+12	−6	+2	−9	+6	−1
95	+3	+3	+11	−5	−14	−17	+28	+3	+14	−39	+11	−14	+39	−25	+5
941	+9	+24	−24	+5	+28	+2	−16	+9	−28	+57	−24	+8	−42	+29	−5
932	+21	+12	−20	+5	+28	−11	−32	+27	+5	+30	−26	+4	−27	+25	−5
931^2	$\boxed{-22}$	−23	+24	−5	−42	+18	+32	−25	+9	−49	+27	−10	+42	−29	+5
92^21	−23	$\boxed{-16}$	+22	−5	−42	+26	+16	−29	+9	−42	+28	−4	+33	−27	+5
921^3	+24	+22	$\boxed{-24}$	+5	+56	−28	−24	+29	−12	+58	−29	+8	−41	+29	−5
91^5	−5	−5	+5	$\boxed{-1}$	−14	+6	+6	−6	+3	−12	+6	−2	+9	−6	+1
86	−42	−42	+56	−14	$\boxed{+34}$	−20	−20	+6	−10	+12	+8	+2	+12	−22	+6
851	+13	+26	−28	+6	−20	$\boxed{+10}$	−8	+5	−4	+41	−20	+12	−52	+35	−6
842	+32	+16	−24	+6	−20	−8	$\boxed{-8}$	+16	−4	+48	−32	0	−32	+30	−6
841^2	−25	−29	+29	−6	+6	+5	+16	$\boxed{-17}$	+18	−67	+33	−10	+51	−35	+6
83^2	+9	+9	−12	+3	−10	−4	−4	+18	$\boxed{-5}$	+9	−12	+6	−18	+15	−3
8321	−49	−42	+53	−12	+12	+41	+48	−67	+9	$\boxed{-38}$	+61	−16	+38	−65	+12
831^3	+27	+28	−29	+6	+8	−20	−32	+33	−12	+61	$\boxed{-33}$	+12	−51	+35	−6
82^3	−10	−4	+8	−2	+2	+12	0	−10	+6	−16	+12	$\boxed{0}$	+10	−10	+2
82^21^2	+42	+33	−41	+9	+12	−52	−32	+51	−18	+83	−51	+10	$\boxed{-65}$	+50	−9
821^4	−29	−27	+29	−6	−22	+35	+30	−35	+15	−65	+35	−10	+50	$\boxed{-35}$	+6
81^6	+5	+5	−5	+1	+6	−6	−6	+6	−3	+12	−6	+2	−9	+6	$\boxed{-1}$
7^2	−21	−21	+28	−7	−7	+14	+14	−21	+7	−42	+28	−7	+42	−35	+7
761	+58	+71	−78	+15	−20	+5	−8	+10	−4	+46	−25	+12	−57	+40	−7
752	+27	−9	−19	+7	+28	−11	−32	+27	−28	+75	−43	+4	−39	+35	−7
751^2	−29	−31	+33	−7	+6	+5	+24	−21	+18	−75	+37	−14	+61	−41	+7
743	+11	+31	−28	+7	+28	−56	−16	+44	+5	+29	−28	+8	−42	+35	−7
7421	−54	−52	+62	−14	−36	+49	+56	−71	+27	−122	+72	−12	+97	−76	+14
741^3	+30	+34	−34	+7	+8	−20	−32	+33	−21	+79	−39	+12	−60	+41	−7
73^21	−24	−27	+31	−7	−18	+47	+20	−36	0	−42	+31	−14	+51	−38	+7
732^2	−31	−16	+28	−7	−42	+39	+16	−37	+9	−46	+38	−4	+37	−35	+7
7321^2	+90	+82	−96	+21	+72	−112	−38	+114	−27	+178	−111	+32	−157	+117	−21
731^4	−32	−33	+34	−7	−22	+35	+38	−39	+15	−78	+39	−14	+60	−41	+7
72^31	+33	+20	−30	+7	+40	−38	−16	+39	−15	+58	−40	+4	−43	+37	−7
72^21^3	−66	−55	+65	−14	−68	+80	+56	−80	+30	−136	+80	−18	+106	−79	+14
6^22	+36	+18	−32	+8	−34	+20	+32	−12	+10	−36	+4	−14	+30	−8	0
6^21^2	−37	−38	+39	−8	+27	−19	−12	+11	−3	+8	−3	+1	−3	+1	0

	7^2	761	752	751^2	743	7421	741^3	73^21	732^2	7321^2	731^4	72^31	72^21^3	6^22	6^21^2
14	+ 7	− 28	− 28	+ 42	− 28	+ 84	− 56	+ 42	+ 42	−168	+ 70	− 56	+140	− 14	+ 21
13.1	− 7	+ 15	+ 28	− 16	+ 28	− 58	+ 17	− 29	− 42	+ 90	− 18	+ 43	− 62	+ 14	− 8
12.2	− 7	+ 28	+ 4	− 30	+ 28	− 36	+ 32	− 42	+ 6	+ 72	− 34	− 16	− 8	+ 2	− 15
12.1^2	+ 7	− 15	− 16	+ 16	− 28	+ 34	− 17	+ 29	+ 18	− 54	+ 18	− 7	+ 14	− 8	+ 8
11.3	− 7	+ 28	+ 28	− 42	− 5	− 51	+ 45	+ 24	− 9	+ 3	− 15	+ 23	− 30	+ 14	− 21
11.21	+ 14	− 43	− 32	+ 46	− 23	+ 83	− 49	+ 5	+ 8	− 41	+ 19	− 16	+ 37	− 16	+ 23
11.1^3	− 7	+ 15	+ 16	− 16	+ 17	− 34	+ 17	− 7	− 7	+ 21	− 7	+ 7	− 14	+ 8	− 8
10.4	− 7	+ 28	+ 28	− 42	− 12	− 4	+ 16	− 2	− 22	+ 48	− 20	+ 16	− 40	+ 14	− 21
10.31	+ 14	− 43	− 56	+ 58	+ 17	+ 29	− 22	− 5	+ 31	− 63	+ 23	− 26	+ 52	− 28	+ 29
10.2^2	+ 7	− 28	− 4	+ 30	− 8	− 4	− 12	+ 22	+ 4	− 32	+ 14	− 4	+ 18	− 2	+ 15
10.21^2	− 21	+ 58	+ 48	− 62	+ 11	− 37	+ 26	− 24	− 21	+ 75	− 27	+ 23	− 51	+ 24	− 31
10.1^4	+ 7	− 15	− 16	+ 16	− 7	+ 14	− 7	+ 7	+ 7	− 21	+ 7	− 7	+ 14	− 8	+ 8
95	− 7	+ 28	− 17	+ 8	+ 28	− 39	+ 11	− 42	+ 3	+ 78	− 25	+ 11	− 50	+ 14	− 21
941	+ 14	− 43	− 11	+ 18	− 16	+ 53	− 24	+ 31	+ 19	− 84	+ 29	− 32	+ 66	− 28	+ 29
932	+ 14	− 56	+ 13	+ 27	− 23	+ 42	− 32	+ 18	+ 12	− 66	+ 31	− 16	+ 47	− 16	+ 36
931^2	− 21	+ 58	+ 27	− 29	+ 11	− 54	+ 30	− 24	− 31	+ 90	− 32	+ 33	− 66	+ 36	− 37
92^21	− 21	+ 71	− 9	− 31	+ 31	− 52	+ 34	− 27	− 16	+ 82	− 33	+ 20	− 55	+ 18	− 38
921^3	+ 28	− 78	− 19	+ 33	− 28	+ 62	− 34	+ 31	+ 28	− 96	+ 34	− 30	+ 65	− 32	+ 39
91^5	− 7	+ 15	+ 7	− 7	+ 7	− 14	+ 7	− 7	− 7	+ 21	− 7	+ 7	− 14	+ 8	− 8
86	− 7	− 20	+ 28	+ 6	+ 28	− 36	+ 8	− 18	− 42	+ 72	− 22	+ 40	− 68	− 34	+ 27
851	+ 14	+ 5	− 11	+ 5	− 56	+ 49	− 20	+ 47	+ 39	−112	+ 35	− 38	+ 80	+ 20	− 19
842	+ 14	− 8	− 32	+ 24	− 16	+ 56	− 32	+ 20	+ 16	− 88	+ 38	− 16	+ 56	+ 32	− 12
841^2	− 21	+ 10	+ 27	− 21	+ 44	− 71	+ 33	− 36	− 37	+114	− 39	+ 39	− 80	− 12	+ 11
83^2	+ 7	− 4	− 28	+ 18	+ 5	+ 27	− 21	0	+ 9	− 27	+ 15	− 15	+ 30	+ 10	− 8
8321	− 42	+ 46	+ 75	− 75	+ 29	−122	+ 79	− 42	− 46	+178	− 73	+ 58	−136	− 36	+ 8
831^3	+ 28	− 25	− 43	+ 37	− 28	+ 72	− 39	+ 31	+ 38	−111	+ 39	− 40	+ 80	+ 4	− 3
82^3	− 7	+ 12	+ 4	− 14	+ 8	− 12	+ 12	− 14	− 4	+ 32	− 14	+ 4	− 18	− 14	+ 1
82^21^2	+ 42	− 57	− 39	+ 61	− 42	+ 97	− 60	+ 51	+ 37	−157	+ 60	− 48	+106	+ 30	− 8
821^4	− 35	+ 40	+ 35	− 41	+ 35	− 76	+ 41	− 38	− 35	+117	− 41	+ 37	− 79	− 8	+ 1
81^6	+ 7	− 7	− 7	+ 7	− 7	+ 14	− 7	+ 7	+ 7	− 21	+ 7	− 7	+ 14	0	0
7^2	$\boxed{+21}$	− 35	− 35	+ 28	− 35	+ 56	− 21	+ 28	+ 28	− 63	+ 14	− 21	+ 28	+ 7	+ 14
761	− 35	$\boxed{+61}$	+ 42	− 46	+ 42	− 60	+ 31	− 30	− 14	+ 48	− 16	+ 1	− 3	+ 20	− 26
752	− 35	+ 42	$\boxed{+41}$	− 36	+ 42	− 51	+ 20	− 14	− 37	+ 39	− 4	+ 33	− 33	− 16	− 13
751^2	+ 28	− 46	− 36	$\boxed{+35}$	− 14	+ 36	− 19	+ 1	+ 6	− 9	+ 3	− 4	+ 4	− 12	+ 18
743	− 35	+ 42	+ 42	− 14	$\boxed{+31}$	− 57	+ 9	− 36	− 25	+ 75	− 14	+ 17	− 28	− 28	− 7
7421	+ 56	− 60	− 51	+ 36	− 57	$\boxed{+60}$	− 14	+ 35	+ 46	− 74	+ 12	− 34	+ 37	+ 12	+ 16
741^3	− 21	+ 31	+ 20	− 19	+ 9	− 14	$\boxed{+6}$	− 6	− 5	+ 12	− 3	+ 3	− 4	+ 4	− 10
73^21	+ 28	− 30	− 14	+ 1	− 36	+ 35	− 6	$\boxed{+40}$	+ 16	− 57	+ 11	− 8	+ 16	+ 18	+ 2
732^2	+ 28	− 14	− 37	+ 6	− 25	+ 46	− 5	+ 16	$\boxed{+33}$	− 49	+ 4	− 29	+ 33	+ 18	− 2
7321^2	− 63	+ 48	+ 39	− 9	+ 75	− 74	+ 12	− 57	− 49	$\boxed{+98}$	− 15	+ 33	− 41	− 24	− 3
731^4	+ 14	− 16	− 4	+ 3	− 14	+ 12	− 3	+ 11	+ 4	− 15	$\boxed{+3}$	− 2	+ 4	+ 4	+ 2
72^31	− 21	+ 1	+ 33	− 4	+ 17	− 34	+ 3	− 8	− 29	+ 33	− 2	$\boxed{+25}$	− 25	− 4	+ 2
72^21^3	+ 28	− 3	− 33	+ 4	− 28	+ 37	− 4	+ 16	+ 33	− 41	+ 4	− 25	$\boxed{+25}$	+ 2	− 1
6^22	+ 7	+ 20	− 16	− 12	− 28	+ 12	+ 4	+ 18	+ 18	− 24	+ 4	− 4	+ 2	$\boxed{+4}$	− 12
6^21^2	+ 14	− 26	− 13	+ 18	− 7	+ 16	− 10	+ 2	− 2	− 3	+ 2	+ 2	− 1	− 12	$\boxed{+12}$

	653	6521	651^3	64^2	6431	642^2	6421^2	641^4	63^22	63^21^2	632^21	62^4	5^24	5^231	5^22^2
14	− 28	+ 84	− 56	− 14	+ 84	+ 42	−168	+ 70	+ 42	− 84	−168	− 14	− 14	+ 42	+ 21
13.1	+ 28	− 58	+ 17	+ 14	− 58	− 42	+ 90	− 18	− 42	+ 45	+129	+ 14	+ 14	− 29	− 21
12.2	+ 28	− 36	+ 32	+ 14	− 84	+ 6	+ 72	− 34	− 18	+ 72	+ 24	− 10	+ 14	− 42	+ 3
12.1^2	− 28	+ 34	− 17	− 14	+ 58	+ 18	− 54	+ 18	+ 30	− 45	− 57	− 2	− 14	+ 29	+ 9
11.3	− 5	− 51	+ 45	+ 14	− 18	− 42	+102	− 48	+ 24	− 15	+ 3	+ 14	+ 14	− 9	− 21
11.21	− 28	+ 88	− 49	− 28	+ 76	+ 36	−140	+ 52	− 6	− 18	− 32	− 4	− 28	+ 38	+ 18
11.1^3	+ 17	− 34	+ 17	+ 14	− 36	− 18	+ 54	− 18	− 8	+ 12	+ 24	+ 2	+ 14	− 18	− 9
10.4	+ 28	− 84	+ 56	− 26	+ 86	+ 18	+ 8	− 20	− 42	+ 4	+ 88	− 6	− 6	− 22	− 11
10.31	− 23	+109	− 62	+ 12	− 14	+ 24	− 62	+ 26	+ 18	− 10	− 82	− 8	− 8	+ 38	+ 32
10.2^2	− 28	+ 36	− 32	+ 6	+ 24	− 16	− 12	+ 14	+ 18	− 32	− 24	0	− 4	+ 32	+ 2
10.21^2	+ 51	−117	+ 66	+ 2	− 44	− 14	+ 74	− 30	− 24	+ 43	+ 79	+ 6	+ 22	− 62	− 27
10.1^4	− 17	+ 34	− 17	− 4	+ 16	+ 8	− 24	+ 8	+ 8	− 12	− 24	− 2	− 9	+ 18	+ 9
95	− 17	+ 6	+ 11	+ 14	− 39	− 42	+ 78	− 25	+ 8	+ 39	+ 33	+ 14	− 31	+ 48	+ 24
941	− 11	+ 52	− 28	+ 12	+ 18	+ 24	− 80	+ 29	+ 39	− 40	−100	− 8	+ 37	− 39	− 18
932	+ 22	− 3	− 32	− 28	+ 57	+ 36	− 84	+ 37	+ 3	− 39	− 54	− 4	+ 17	− 39	− 27
931^2	+ 6	− 53	+ 34	+ 2	− 35	− 42	+ 98	− 35	− 30	+ 46	+108	+ 10	− 23	+ 28	+ 4
92^21	+ 6	− 29	+ 36	+ 22	− 73	− 20	+ 98	− 39	− 21	+ 50	+ 74	+ 4	− 13	+ 20	+ 25
921^3	− 23	+ 61	− 38	− 16	+ 71	+ 32	−110	+ 39	+ 32	− 55	−108	− 8	+ 9	− 10	− 9
91^5	+ 8	− 16	+ 8	+ 4	− 16	− 8	+ 24	− 8	− 8	+ 12	+ 24	+ 2	0	0	0
86	− 20	+ 60	− 40	− 10	+ 12	+ 54	− 72	+ 26	+ 30	− 12	− 72	− 14	+ 14	+ 6	− 21
851	+ 37	− 52	+ 28	− 4	+ 1	− 12	+ 32	− 13	− 33	+ 12	+ 38	0	+ 17	− 27	− 3
842	− 8	− 24	+ 8	+ 36	− 48	− 56	+ 64	− 10	− 12	+ 20	+ 64	+ 12	− 8	+ 16	+ 8
841^2	− 9	+ 18	− 11	− 22	+ 25	+ 22	− 34	+ 9	+ 8	− 11	− 11	− 2	− 23	+ 18	+ 4
83^2	+ 29	− 21	+ 3	− 2	− 30	− 6	+ 18	0	− 24	+ 27	+ 45	0	− 14	− 15	+ 21
8321	− 72	+ 59	− 9	− 4	+ 73	+ 32	− 58	+ 5	+ 57	− 53	−112	− 8	+ 19	+ 24	− 23
831^3	+ 25	− 17	+ 5	+ 8	− 25	− 4	+ 16	− 3	− 10	+ 14	+ 17	0	+ 9	− 18	+ 5
82^3	+ 12	+ 12	0	− 14	+ 8	+ 16	− 20	+ 2	+ 6	0	− 24	− 4	+ 4	− 16	− 2
82^21^2	+ 15	− 30	+ 2	+ 12	− 27	− 30	+ 36	− 3	− 27	+ 15	+ 63	+ 8	− 9	+ 10	+ 2
821^4	− 8	+ 9	− 1	− 4	+ 9	+ 8	− 10	+ 1	+ 8	− 5	− 17	− 2	0	0	0
81^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7^2	+ 14	+ 7	− 21	+ 7	+ 7	− 21	− 14	+ 14	− 21	− 7	+ 35	+ 7	+ 7	− 21	+ 14
761	− 8	− 58	+ 37	− 4	− 10	− 12	+ 52	− 22	+ 12	+ 2	− 1	0	− 28	+ 28	− 7
752	− 11	+ 2	+ 20	− 28	+ 25	+ 36	− 24	− 4	+ 15	− 13	− 29	− 4	+ 17	− 6	− 6
751^2	− 9	+ 39	− 25	+ 18	− 3	− 6	− 13	+ 9	+ 3	− 2	− 2	+ 2	− 3	− 2	+ 8
743	− 23	+ 37	− 3	+ 12	− 32	+ 24	+ 2	− 2	+ 18	+ 25	− 35	− 8	− 8	+ 31	− 17
7421	+ 42	− 27	− 8	− 20	+ 29	− 4	− 12	+ 6	− 21	− 11	+ 26	0	− 1	− 15	+ 15
741^3	− 8	− 5	+ 8	+ 8	− 17	− 4	+ 15	− 5	+ 5	+ 6	− 6	0	+ 9	0	− 9
73^21	− 6	− 11	+ 3	− 10	+ 37	− 18	− 5	+ 2	+ 6	− 31	+ 2	+ 8	+ 22	− 18	− 4
732^2	+ 6	− 5	+ 1	+ 22	− 25	− 20	+ 12	− 2	− 21	+ 15	+ 29	+ 4	− 18	+ 7	+ 4
7321^2	+ 15	+ 6	0	0	− 36	+ 24	+ 5	− 3	− 3	+ 27	− 7	− 8	− 18	+ 10	+ 4
731^4	− 8	+ 4	− 2	− 4	+ 16	− 4	− 6	+ 2	+ 2	− 9	0	+ 2	0	0	0
72^31	− 18	+ 3	− 1	− 8	+ 23	+ 4	− 8	+ 2	+ 15	− 15	− 15	0	+ 9	− 4	− 2
72^21^3	+ 8	− 3	+ 1	+ 4	− 9	− 2	+ 4	− 1	− 5	+ 5	+ 5	0	0	0	0
6^22	+ 20	− 12	+ 16	+ 10	− 12	− 6	+ 12	− 8	− 6	0	0	+ 2	− 14	− 6	+ 9
6^21^2	− 6	+ 27	− 16	− 3	+ 3	+ 9	− 20	+ 8	− 3	+ 5	+ 2	− 1	+ 21	− 2	− 1

	5^221^2	54^21	5432	5431^2	53^3	4^32	721^5	6321^3	542^21	4^23^2	4^31^2	53^221	4^2321	43^31	532^3
14	− 84	+ 42	+ 84	−168	+ 14	+ 14	− 84	+280	−168	+ 21	− 28	−168	−168	− 56	− 56
13.1	+ 45	− 29	− 84	+ 90	− 14	− 14	+ 19	−124	+129	− 21	+ 15	+129	+129	+ 43	+ 56
12.2	+ 36	− 42	− 36	+144	− 14	− 6	+ 12	−112	+ 24	− 21	+ 24	+ 96	+ 96	+ 56	− 16
12.1^2	− 27	+ 29	+ 60	− 90	+ 14	+ 10	− 7	+ 76	− 57	+ 21	− 15	− 93	− 93	− 43	− 20
11.3	+ 51	− 42	− 18	+ 69	+ 19	− 14	+ 18	− 27	+102	+ 12	+ 28	− 63	+ 36	− 43	+ 28
11.21	− 70	+ 71	+ 54	−135	− 5	+ 20	− 20	+ 71	−131	+ 9	− 39	− 16	−115	0	− 7
11.1^3	+ 27	− 29	− 38	+ 57	− 8	− 10	+ 7	− 82	+ 57	− 10	+ 15	+ 27	+ 60	+ 10	+ 9
10.4	+ 64	+ 38	− 4	− 82	− 14	+ 26	+ 24	− 80	+ 8	+ 19	− 33	+ 88	−112	− 24	+ 16
10.31	− 91	− 9	+ 22	− 19	− 5	− 12	− 27	+101	− 71	− 31	+ 17	− 46	+ 85	+ 50	− 39
10.2^2	− 36	+ 2	− 4	− 44	+ 14	− 14	− 12	+ 52	+ 16	+ 1	+ 6	− 56	+ 44	− 16	− 4
10.21^2	+ 97	− 20	− 34	+ 85	− 9	+ 10	+ 27	−117	+ 68	+ 10	− 6	+ 89	− 42	− 7	+ 27
10.1^4	− 27	+ 9	+ 18	− 27	+ 3	0	− 7	+ 32	− 27	0	0	− 27	0	0	− 9
95	− 51	+ 48	+ 51	−102	+ 1	− 14	+ 30	−100	− 57	− 21	− 17	− 57	+ 33	+ 41	− 34
941	+ 26	− 63	− 47	+104	+ 18	− 12	− 34	+182	+ 34	+ 2	+ 26	+ 8	+ 46	− 16	+ 18
932	+ 48	− 6	− 27	+ 30	− 20	+ 20	− 30	+112	+ 45	+ 9	− 7	+ 84	− 51	+ 2	+ 29
931^2	− 17	+ 34	+ 26	− 53	+ 6	+ 2	+ 34	−141	− 16	+ 10	− 11	− 32	− 28	− 10	− 4
92^21	− 29	− 1	+ 31	− 10	+ 6	− 6	+ 32	−182	− 47	− 10	+ 6	− 45	+ 26	+ 7	− 25
921^3	+ 11	− 5	− 18	+ 11	− 3	0	− 34	+149	+ 19	0	0	+ 19	0	0	+ 9
91^5	0	0	0	0	0	0	+ 7	− 32	0	0	0	0	0	0	0
86	− 12	− 18	− 36	+ 24	+ 10	+ 10	+ 36	+ 56	+ 24	− 9	+ 4	− 24	+ 24	+ 8	+ 40
851	+ 22	− 23	− 15	+ 36	− 11	+ 4	− 41	− 32	+ 32	+ 30	+ 6	+ 40	− 38	− 36	− 6
842	− 4	− 20	+ 56	0	+ 4	− 12	− 36	− 48	− 16	− 10	+ 16	− 56	0	+ 16	− 16
841^2	− 15	+ 38	+ 8	− 38	− 3	+ 10	+ 41	+ 16	− 17	− 11	− 15	+ 5	+ 3	+ 11	+ 2
83^2	− 3	+ 30	− 6	+ 3	+ 5	+ 2	− 18	− 45	− 30	0	− 16	+ 15	0	− 5	− 15
8321	+ 1	− 28	− 17	− 12	+ 6	− 12	+ 77	+ 92	+ 37	+ 1	+ 17	− 22	+ 21	− 4	+ 17
831^3	+ 6	− 9	− 4	+ 20	− 3	0	− 41	− 19	− 1	0	0	+ 13	0	0	− 5
82^3	+ 4	+ 6	− 12	+ 12	− 6	+ 6	+ 12	+ 12	0	+ 3	− 6	+ 24	− 12	0	+ 4
82^21^2	− 4	+ 5	+ 11	− 11	+ 3	0	− 59	− 39	− 5	0	0	− 12	0	0	− 2
821^4	0	0	0	0	0	0	+ 41	+ 11	0	0	0	0	0	0	0
81^6	0	0	0	0	0	0	− 7	0	0	0	0	0	0	0	0
7^2	− 7	− 21	+ 7	+ 35	− 7	− 7	− 7	+ 7	− 14	+ 14	+ 14	+ 35	− 14	− 21	− 21
761	+ 23	+ 47	+ 22	− 58	+ 4	+ 4	+ 1	− 16	+ 1	− 19	− 19	− 28	+ 1	+ 26	+ 2
752	+ 8	− 6	− 48	+ 21	+ 13	+ 20	+ 7	+ 23	+ 19	− 7	− 7	− 11	− 3	+ 1	+ 8
751^2	− 16	− 6	+ 18	− 2	− 3	− 14	− 1	+ 5	− 24	− 2	+ 9	+ 11	+ 12	+ 2	− 2
743	− 17	+ 4	+ 8	− 23	− 5	− 12	+ 7	− 19	+ 2	+ 4	+ 4	− 11	+ 20	− 3	+ 17
7421	− 1	+ 12	− 21	− 4	− 12	+ 4	− 8	+ 6	+ 1	+ 13	− 3	+ 43	− 15	− 7	− 9
741^3	+ 9	− 9	0	+ 9	+ 6	0	+ 1	− 5	+ 9	0	0	− 18	0	0	+ 3
73^21	+ 17	− 21	− 2	+ 23	0	+ 10	− 4	+ 28	+ 10	− 4	− 1	− 4	− 8	+ 4	− 2
732^2	− 5	+ 4	+ 17	− 7	+ 6	− 6	− 7	− 17	− 5	− 3	+ 1	− 14	+ 7	0	− 4
7321^2	− 11	+ 14	+ 4	− 13	− 3	0	+ 9	− 23	− 12	0	0	+ 7	0	0	+ 2
731^4	0	0	0	0	0	0	− 1	+ 8	0	0	0	0	0	0	0
72^31	+ 4	− 5	− 5	+ 5	0	0	+ 5	+ 15	+ 5	0	0	0	0	0	0
72^21^3	0	0	0	0	0	0	− 5	− 5	0	0	0	0	0	0	0
6^22	0	+ 18	+ 12	− 12	− 10	− 14	0	+ 4	− 24	+ 9	− 2	+ 24	+ 12	− 8	− 4
6^21^2	− 7	− 26	− 17	+ 23	+ 3	+ 5	0	− 3	+ 18	+ 5	+ 5	− 7	− 5	− 5	+ 1

	$4^2 2^3$	$43^2 2^2$	$3^4 2$	$5^2 1^4$	5421^3	$53^2 1^3$	$4^2 31^3$	$62^3 1^2$	$532^2 1^2$	$4^2 2^2 1^2$	$43^2 21^2$	$3^4 1^2$	$52^4 1$	$432^3 1$	$3^3 2^2 1$
14	− 28	− 84	− 14	+ 35	+280	+140	+140	+140	+420	+210	+420	+ 35	+ 70	+280	− +140
13.1	+ 28	+ 84	+ 14	− 9	−124	− 62	− 62	− 88	−264	−132	−264	− 22	− 57	−228	− 114
12.2	− 8	+ 12	+ 8	− 17	−112	−104	−104	+ 16	− 96	− 48	−240	− 32	+ 26	+ 8	− 44
12.1^2	− 10	− 48	− 11	+ 9	+ 76	+ 62	+ 62	+ 16	+120	+ 60	+192	+ 22	+ 9	+ 84	− + 66
11.3	+ 28	− 15	− 19	− 24	−159	+ 3	− 63	− 41	− 24	−111	+ 75	+ 31	− 37	− 49	− + 91
11.21	− 20	+ 8	+ 11	+ 26	+203	+ 34	+100	+ 39	+ 96	+147	+ 75	− 12	+ 20	+ 55	− 51
11.1^3	+ 10	+ 15	0	− 9	− 76	− 18	− 40	− 16	− 54	− 60	− 60	0	− 9	− 40	0
10.4	− 32	+ 4	+ 14	− 30	− 40	− 20	+ 80	− 40	−160	+100	+100	+ 5	− 10	+ 40	− 60
10.31	+ 4	+ 11	+ 5	+ 33	+103	+ 29	− 50	+ 59	+178	− 37	−121	− 19	+ 34	− 13	− 12
10.2^2	+ 18	+ 8	− 8	+ 17	+ 32	+ 44	− 6	+ 14	+ 66	− 57	0	+ 12	+ 4	− 48	+ 24
10.21^2	− 10	− 15	0	− 35	−119	− 66	+ 13	− 55	−186	+ 33	+ 33	0	− 29	+ 31	0
10.1^4	0	0	0	+ 9	+ 36	+ 18	0	+ 16	+ 54	0	0	0	+ 9	0	0
95	+ 28	− 6	− 1	+ 10	+ 80	+ 40	+ 40	− 50	+120	− 30	− 60	− 20	+ 20	− 10	+ 10
941	+ 4	+ 2	− 13	− 6	− 52	− 26	− 54	+ 74	− 44	− 22	− 16	+ 14	− 5	− 10	+ 30
932	− 20	− 15	+ 8	− 4	− 44	− 25	+ 14	+ 43	−123	+ 33	+ 21	− 5	− 25	+ 41	− 17
931^2	+ 6	+ 1	0	+ 3	+ 28	+ 17	+ 24	− 75	+ 35	+ 4	+ 28	9	+ 2	− 8	0
$92^2 1$	+ 2	+ 7	0	+ 2	+ 26	+ 12	− 13	− 53	+ 81	− 9	− 21	0	+ 21	− 7	0
921^3	0	0	0	− 1	− 12	− 6	0	+ 71	− 30	0	0	0	− 7	0	0
91^5	0	0	0	0	0	0	0	− 16	0	0	0	0	0	0	0
86	− 28	+ 12	− 10	+ 18	+ 8	+ 4	− 20	+ 44	− 12	+ 18	− 36	+ 1	− 26	− 8	+ 28
851	0	− 6	+ 11	− 9	− 44	− 22	+ 2	− 6	− 24	− 6	+ 60	+ 6	+ 8	+ 6	− 24
842	+ 16	− 8	+ 2	− 1	− 8	+ 12	− 16	− 48	+ 56	− 16	+ 12	− 9	+ 4	0	0
841^2	− 10	+ 6	0	+ 5	+ 32	+ 8	+ 12	+ 6	− 4	+ 18	− 24	0	− 2	− 2	0
83^2	0	+ 15	− 5	0	+ 15	− 15	+ 15	− 15	0	+ 15	− 15	+ 5	+ 15	− 15	+ 5
8321	+ 4	− 7	0	+ 1	− 14	+ 22	− 16	+ 51	+ 1	− 23	+ 9	0	− 15	+ 5	0
831^3	0	0	0	− 2	− 8	− 11	0	− 5	− 5	0	0	0	+ 5	0	0
82^3	− 2	0	0	− 1	0	− 12	+ 6	+ 18	− 18	+ 9	0	0	0	0	0
$82^2 1^2$	0	0	0	+ 1	+ 5	+ 6	0	− 36	+ 9	0	0	0	0	0	0
821^4	0	0	0	0	0	0	0	+ 9	0	0	0	0	0	0	0
81^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7^2	+ 14	− 7	+ 7	+ 7	+ 7	− 21	− 21	− 21	− 14	− 7	+ 35	+ 7	+ 14	+ 7	− 21
761	0	+ 2	− 4	− 11	− 10	+ 23	+ 26	+ 2	+ 10	+ 2	− 22	− 7	− 1	− 2	− 9
752	− 20	+ 22	− 7	− 7	− 17	+ 1	+ 1	+ 6	+ 4	+ 22	− 15	+ 3	− 4	− 12	− 6
751^2	+ 10	− 9	0	+ 7	+ 24	− 5	− 8	− 3	− 5	− 12	0	0	+ 2	+ 8	0
743	+ 4	− 17	+ 5	+ 5	+ 17	+ 3	− 3	+ 25	+ 2	− 17	+ 7	− 1	− 9	+ 9	− 3
7421	0	+ 3	0	+ 2	+ 4	− 6	+ 5	− 9	− 24	+ 3	+ 6	0	+ 9	− 3	0
741^3	0	0	0	− 3	− 12	+ 3	0	+ 3	+ 9	0	0	0	− 3	0	0
$73^2 1$	− 4	+ 2	0	− 5	− 20	0	0	− 16	+ 4	+ 8	− 4	0	0	0	0
732^2	+ 2	0	0	+ 1	+ 5	+ 2	− 1	− 8	+ 8	− 4	0	0	0	0	0
7321^2	0	0	0	+ 3	+ 14	− 1	0	+ 16	− 4	0	0	0	0	0	0
731^4	0	0	0	0	0	0	0	− 4	0	0	0	0	0	0	0
$72^3 1$	0	0	0	− 1	− 5	0	0	0	0	0	0	0	0	0	0
$72^2 1^3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$6^2 2$	+ 10	− 12	+ 4	− 4	+ 16	− 4	+ 2	− 2	− 6	− 9	0	+ 2	+ 2	+ 8	− 4
$6^2 1^2$	− 5	+ 5	0	+ 4	− 9	− 2	− 5	+ 1	+ 4	+ 5	+ 5	0	− 1	− 5	0

	42^5	3^22^4	631^5	541^5	62^21^4	5321^4	4^221^4	43^21^4	52^31^3	432^21^3	3^321^3	42^41^2	$3^22^31^2$	32^51	2^7
14	+ 14	+ 35	− 84	− 84	−210	−420	−210	−210	−280	−840	−280	−210	−420	− 84	− 2
13.1	− 14	− 35	+ 19	+ 19	+ 80	+160	+ 80	+ 80	+150	+450	+150	+145	+290	+ 71	+ 2
12.2	+ 10	+ 13	+ 36	+ 36	+ 18	+156	+ 78	+138	− 8	+216	+152	− 42	+ 36	− 36	− 2
12.1^2	+ 2	+ 11	− 19	− 19	− 20	−100	− 50	− 80	− 30	−210	−110	− 25	−110	− 11	0
11.3	− 14	− 2	+ 18	+ 51	+ 45	+ 57	+111	+ 12	+ 71	+ 81	−127	+ 78	−108	+ 18	+ 2
11.21	+ 4	− 11	− 22	− 55	− 54	−107	−136	− 53	− 76	−204	+ 50	− 59	+ 70	+ 9	0
11.1^3	− 2	0	+ 8	+ 19	+ 20	+ 45	+ 50	+ 25	+ 30	+100	0	+ 25	0	0	0
10.4	+ 14	− 5	+ 24	+ 24	+ 60	+120	− 60	− 60	+ 80	−120	0	− 50	+ 80	− 4	− 2
10.31	0	+ 7	− 27	− 30	− 75	−147	+ 24	+ 48	−111	+ 69	+ 47	+ 7	− 7	− 7	0
10.2^2	− 4	+ 2	− 16	− 16	− 28	− 76	+ 22	− 8	− 32	+ 64	− 32	+ 32	− 16	0	0
10.21^2	+ 2	0	+ 31	+ 34	+ 74	+167	− 14	− 7	+106	− 46	0	− 16	0	0	0
10.1^4	0	0	− 8	− 9	− 20	− 45	0	0	− 30	0	0	0	0	0	0
95	− 14	+ 10	+ 30	− 15	+ 75	− 75	− 15	− 15	− 50	+ 30	+ 40	+ 30	− 30	− 6	+ 2
941	0	− 5	− 34	+ 11	− 95	+ 35	+ 25	+ 25	+ 10	+ 30	− 30	− 5	− 10	+ 5	0
932	+ 4	− 2	− 36	+ 3	− 72	+ 57	− 9	− 6	+ 60	− 36	+ 11	− 18	+ 9	0	0
931^2	− 2	0	+ 37	− 5	+ 95	− 23	− 10	− 13	− 9	− 12	0	+ 9	0	0	0
92^21	0	0	+ 38	− 1	+ 82	− 33	+ 6	+ 7	− 42	+ 14	0	0	0	0	0
921^3	0	0	− 39	+ 1	− 94	+ 13	0	0	+ 14	0	0	0	0	0	0
91^5	0	0	+ 8	0	+ 20	0	0	0	0	0	0	0	0	0	0
86	+ 14	− 13	− 12	− 12	− 30	− 12	− 6	+ 18	+ 24	+ 24	− 8	− 18	− 12	+ 12	− 2
851	0	+ 3	+ 7	+ 12	+ 5	+ 27	+ 11	− 13	− 4	− 24	− 2	+ 3	+ 12	− 3	0
842	− 4	+ 2	+ 4	+ 4	+ 34	− 20	+ 14	+ 2	− 8	− 16	+ 8	+ 8	− 4	0	0
841^2	+ 2	0	− 3	− 8	− 5	− 7	− 15	+ 3	+ 4	+ 12	0	− 4	0	0	0
83^2	0	0	+ 6	− 3	+ 15	+ 15	− 15	0	− 15	+ 15	− 5	0	0	0	0
8321	0	0	− 11	+ 2	− 39	− 19	+ 17	− 1	+ 15	− 5	0	0	0	0	0
831^3	0	0	+ 3	+ 2	+ 5	+ 10	0	0	− 5	0	0	0	0	0	0
82^3	0	0	0	0	− 12	+ 12	− 6	0	0	0	0	0	0	0	0
82^21^2	0	0	+ 3	− 1	+ 24	− 6	0	0	0	0	0	0	0	0	0
821^4	0	0	− 1	0	− 6	0	0	0	0	0	0	0	0	0	0
81^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7^2	− 7	+ 7	− 7	− 7	+ 7	+ 14	+ 7	+ 7	− 7	− 21	− 7	+ 7	+ 14	− 7	+ 1
761	0	− 1	+ 7	+ 7	− 1	− 8	− 4	− 7	+ 1	+ 9	+ 5	− 1	− 5	+ 1	
752	+ 4	− 2	− 3	+ 7	− 2	− 4	− 7	+ 3	+ 2	+ 6	− 3	− 2	+ 1		
751^2	− 2	0	− 2	− 7	+ 1	+ 3	+ 4	+ 2	− 1	− 4	0	+ 1			
743	0	0	+ 7	− 5	− 7	− 2	+ 5	− 1	+ 3	− 3	+ 1				
7421	0	0	− 4	− 2	+ 3	+ 8	− 1	− 2	− 3	+ 1					
741^3	0	0	+ 2	+ 3	− 1	− 3	0	0	+ 1						
73^21	0	0	− 7	+ 5	+ 4	− 1	− 2	+ 1							
732^2	0	0	+ 3	− 1	+ 2	− 2	+ 1								
7321^2	0	0	+ 6	− 3	− 4	+ 1									
731^4	0	0	− 2	0	+ 1										
72^31	0	0	− 3	+ 1											
72^21^3	0	0	+ 1												
6^22	− 2	+ 1													
6^21^2	+ 1														

	71^7	621^6	531^6	4^21^6	52^21^5	4321^5	3^31^5	42^31^4	$3^22^21^4$	32^41^3	2^61^2	61^8	521^7	431^7	42^21^6
14	+14	+98	+98	+49	+294	+588	+98	+490	+735	+490	+49	−14	−112	−112	−392
13.1	−1	−20	−20	−10	−99	−198	−33	−230	−345	−295	−36	+1	+21	+21	+119
12.2	−2	−14	−38	−19	−30	−204	−58	−10	−195	+50	+25	+2	+16	+40	+44
12.1^2	+1	+8	+20	+10	+27	+126	+33	+50	+165	+55	0	−1	−9	−21	−35
11.3	−3	−21	−21	−27	−63	−93	+23	−116	+156	+16	−16	+3	+24	+24	+84
11.21	+3	+23	+25	+29	+74	+149	−8	+180	−65	−30	0	−3	−26	−28	−97
11.1^3	−1	−8	−9	−10	−27	−60	0	−50	0	0	0	+1	+9	+10	+35
10.4	−4	−28	−28	+6	−84	+72	+12	+60	−30	−30	+9	+4	+32	−8	−28
10.31	+4	+31	+31	−3	+102	−39	−15	−14	−21	+14	0	−4	−35	+5	+7
10.2^2	+2	+14	+18	−1	+40	−16	+8	−40	+20	0	0	−2	−16	0	+16
10.21^2	−4	−31	−35	+1	−101	+15	0	+20	0	0	0	+4	+35	−1	−8
10.1^4	+1	+8	+9	0	+27	0	0	0	0	0	0	−1	−9	0	0
95	−5	−35	+10	+5	+30	+15	−5	−25	−15	+20	−4	+5	−5	−5	+5
941	+5	+39	−6	−5	−6	−27	+3	+5	+15	−5	0	−5	+1	+5	−1
932	+5	+35	−4	+1	−33	+9	−1	+12	−6	0	0	−5	+5	−1	−2
931^2	−5	−39	+3	+2	+6	+12	0	−6	0	0	0	+5	−1	−2	+1
92^21	−5	−37	+2	−1	+21	−7	0	0	0	0	0	+5	−3	+1	
921^3	+5	+30	−1	0	−7	0	0	0	0	0	0	−5	+1		
91^5	−1	−8	0	0	0	0	0	0	0	0	0	+1			
86	−6	+6	+6	+3	−6	−12	−2	+6	+9	−6	+1				
851	+6	−1	−6	−3	+1	+7	+2	−1	−4	+1					
842	+6	−6	+2	−3	+2	+4	−2	−2	+1						
841^2	−6	+1	+2	+3	−1	−3	0	+1							
83^2	+3	−3	−3	+3	+3	−3	+1								
8321	−12	+7	+4	−3	−3	+1									
831^3	+6	−1	−2	0	+1										
82^3	−2	+2	−2	+1											
82^21^2	+9	−4	+1												
821^4	−6	+1													
81^6	+1														

	$3^2 21^6$	$32^3 1^5$	$2^5 1^4$	51^9	421^8	$3^2 1^8$	$32^2 1^7$	$2^4 1^6$	41^{10}	321^9	$2^3 1^8$	31^{11}	$2^2 1^{10}$	21^{12}	1^{14}
14	−392	−784	−196	+ 14	+126	+ 63	+504	+294	− 14	−140	−210	+ 14	+ 77	− 14	+ 1
13.1	+119	+329	+105	− 1	− 22	− 11	−140	−112	+ 1	+ 23	+ 54	− 1	− 12	+ 1	
12.2	+128	+ 40	− 50	− 2	− 18	− 21	− 60	+ 85	+ 2	+ 20	− 10	− 2	+ 1		
12.1^2	− 77	− 77	0	+ 1	+ 10	+ 11	+ 44	0	− 1	− 11	0	+ 1			
11.3	− 48	− 52	+ 20	− 3	− 27	+ 3	+ 24	− 8	+ 3	− 3	+ 1				
11.21	+ 17	+ 27	0	+ 3	+ 29	− 1	− 9	0	− 3	+ 1					
11.1^3	0	0	0	− 1	− 10	0	0	0	+ 1						
10.4	− 8	+ 24	− 6	− 4	+ 4	+ 2	− 4	+ 1							
10.31	+ 14	− 7	0	+ 4	− 1	− 2	+ 1								
10.2^2	− 8	0	0	+ 2	− 2	+ 1									
10.21^2	0	0	0	− 4	+ 1										
10.1^4	0	0	0	+ 1											
95	+ 5	− 5	+ 1												
941	− 3	+ 1													
932	+ 1														

	14	13.1	12.2	12.1^2	11.3	11.21	11.1^3	10.4	10.31	10.2^2	10.21^2	10.1^4	95	941	932
653	−28	+28	+28	−26	−5	−23	+17	+28	−23	−28	+51	−17	−17	−11	+22
6521	+84	−58	−36	+34	−51	+83	−34	−84	+109	+36	−117	+34	+6	+52	−3
651^3	−56	+17	+32	−17	+45	−49	+17	+56	−62	−32	+66	−17	+11	−28	−32
64^2	−14	+14	+14	−14	+14	−28	+14	−26	+12	+6	+2	−4	+14	+12	−28
6431	+84	−58	−84	+58	−18	+76	−36	+36	−14	+24	−44	+16	−39	+13	+57
642^2	+42	−42	+6	+18	−42	+36	−18	+18	+24	−16	−14	+8	−42	+24	+36
6421^2	−168	+90	+72	−54	+102	−140	+54	+8	−62	−12	+74	−24	+78	−80	−84
641^4	+70	−18	−34	+18	−48	+52	−18	−20	+26	+14	−30	+8	−25	+29	+37
63^22	+42	−42	−18	+30	+24	−6	−8	−42	+18	+18	−24	+8	+3	+39	+3
63^21^2	−84	+45	+72	−45	−15	−18	+12	+4	−10	−32	+48	−12	+39	−40	−39
632^21	−168	+129	+24	−57	+3	−32	+24	+88	−82	−24	+79	−24	+33	−100	−54
62^4	−14	+14	−10	−2	+14	−4	+2	−6	−8	0	+6	−2	+14	−8	−4
5^24	−14	+14	+14	−14	+14	−28	+14	−6	−8	−4	+22	−9	−31	+37	+17
5^231	+42	−29	−42	+29	−9	+38	−18	−22	+33	+32	−62	+18	+48	−89	−39
5^22^2	+21	−21	+3	+9	−21	+18	−9	−11	+32	+2	−27	+9	+24	−18	−27
5^221^2	−84	+45	+36	−27	+51	−70	+27	+64	−91	−36	+97	−27	−51	+26	+48
54^21	+42	−29	−42	+29	−42	+71	−29	+88	−9	+2	−20	+9	+48	−63	−6
5432	+84	−84	−36	+60	−18	+54	−38	−4	+22	−4	−34	+18	+51	−47	−27
5431^2	−168	+90	+144	−90	+69	−135	+57	−32	−19	−44	+85	−27	−102	+104	+30
53^3	+14	−14	−14	+14	+19	−5	−3	−14	−5	+14	−9	+3	+1	+13	−20
4^32	+14	−14	−6	+10	−14	+20	−10	+26	−12	−14	+10	0	−14	−12	+20
721^5	−84	+19	+12	−7	+18	−20	+7	+24	−27	−12	+27	−7	+30	−34	−30
6321^3	+280	−124	−112	+76	−27	+71	−32	−80	+101	+52	−117	+32	−100	+132	+112
542^21	−168	+129	+24	−57	+102	−131	+57	+8	−71	+16	+68	−27	−57	+34	+45
4^23^2	+21	−21	−21	+21	+12	+9	−10	+19	−31	+1	+10	0	−21	+2	+9
4^31^2	−28	+15	+24	−15	+28	−39	+15	−32	+17	+6	−6	0	−17	+26	−7
53^221	−168	+129	+96	−93	−63	−16	+27	+88	−46	−56	+89	−27	−57	+8	+84
4^2321	−168	+129	+96	−93	+36	−115	+60	−112	+85	+44	−42	0	+33	+46	−51
43^31	−56	+43	+56	−48	−43	0	+10	−24	+50	−16	−7	0	+41	−16	+2
532^3	−56	+56	−16	−20	+23	−7	+9	+16	−39	−4	+27	−9	−34	+18	+29
4^22^3	−28	+28	−8	−10	+28	−20	+10	−32	+4	+18	−10	0	+28	+4	−20
43^22^2	−84	+84	+12	−48	−15	+8	+15	+4	+11	+8	−15	0	−6	+2	−15
3^42	−14	+14	+8	−11	−19	+11	0	+14	+5	−8	0	0	−1	−13	+8
5^21^4	+35	−9	−17	+9	−24	+26	−9	−30	+33	+17	−35	+9	+10	−6	−4
5421^3	+280	−124	−112	+76	−159	−203	−76	−40	+103	+32	−119	+36	+80	−52	−44
53^21^3	+140	−62	−104	+62	+3	+34	−18	−20	+29	+44	−66	+18	+40	−26	−25
4^231^3	+140	−62	−104	+62	−63	+100	−40	+80	−50	−6	+13	0	+40	−54	+14
62^31^2	+140	−88	+16	+16	−41	+39	−16	−40	+59	+14	−55	+16	−50	+74	+43
532^21^2	+420	−264	−96	+120	−24	+96	−54	−160	+178	+66	−186	+54	+120	−44	−123
$4^22^21^2$	+210	−132	−48	+60	−111	+147	−60	+100	−37	−57	+33	0	−30	−22	+33
43^221^2	+420	−264	−240	+192	+75	+75	−60	+100	−121	0	+33	0	−60	−16	+21
3^41^2	+35	−22	−32	+22	+31	−12	0	+5	−19	+12	0	0	−20	+14	−5
52^41	+70	−57	+26	+9	−37	+20	−9	−10	+34	+4	−29	+9	+20	−5	−25
432^31	+280	−228	+8	+84	−49	+55	−40	+40	−18	−48	+31	0	−10	−10	+41
3^32^21	+140	−114	−44	+66	+91	−51	0	−60	−12	+24	0	0	+10	+30	−17

	931^2	92^21	921^3	91^5	86	851	842	841^2	83^2	8321	831^3	82^3	82^21^2	821^4	81^6
653	+ 6	+ 6	− 23	+ 8	− 20	+ 37	− 8	− 9	+ 29	− 72	+ 25	+ 12	+ 15	− 8	0
6521	− 53	− 29	+ 61	− 16	+ 60	− 52	− 24	+ 18	− 21	+ 59	− 17	+ 12	− 30	+ 9	0
651^3	+ 34	+ 36	− 38	+ 8	− 40	+ 28	+ 8	− 11	+ 3	− 9	+ 5	0	+ 2	− 1	0
64^2	+ 2	+ 22	− 16	+ 4	− 10	− 4	+ 36	− 22	− 2	− 4	+ 8	− 14	+ 12	− 4	0
6431	− 35	− 78	+ 71	− 16	+ 12	+ 1	− 48	+ 25	− 30	+ 73	− 25	+ 8	− 27	+ 9	0
642^2	− 42	− 20	+ 32	− 8	+ 54	− 12	− 56	+ 22	− 6	+ 32	− 4	+ 16	− 30	+ 8	0
6421^2	+ 98	+ 98	−110	+ 24	− 72	+ 32	+ 64	− 34	+ 18	− 58	+ 16	− 20	+ 36	− 10	0
641^4	− 35	− 39	+ 39	− 8	+ 26	− 13	− 10	+ 9	0	+ 5	− 3	+ 2	− 3	+ 1	0
63^22	− 30	− 21	+ 32	− 8	+ 30	− 33	− 12	+ 3	− 24	+ 57	− 10	+ 6	− 27	+ 8	0
63^21^2	+ 46	+ 50	− 55	+ 12	− 12	+ 12	+ 20	− 11	+ 27	− 53	+ 14	0	+ 15	− 5	0
632^21	+103	+ 74	−103	+ 24	− 72	+ 38	+ 64	− 11	+ 45	−112	+ 17	− 24	+ 63	− 17	0
62^4	+ 10	+ 4	− 8	+ 2	− 14	0	+ 12	− 2	0	− 8	0	− 4	+ 8	− 2	0
5^24	− 23	− 13	+ 9	0	+ 14	+ 17	− 8	− 23	− 14	+ 19	+ 9	+ 4	− 9	0	0
5^231	+ 28	+ 20	− 10	0	+ 6	− 27	+ 16	+ 18	− 15	+ 24	− 18	− 16	+ 10	0	0
5^22^2	+ 4	+ 25	− 9	0	− 21	− 3	+ 8	+ 4	+ 21	− 23	+ 5	− 2	+ 2	0	0
5^221^2	− 17	− 29	+ 11	0	− 12	+ 22	− 4	− 15	− 3	+ 1	+ 6	+ 4	− 4	0	0
54^21	+ 34	− 1	− 5	0	− 18	− 23	− 20	+ 38	+ 30	− 28	− 9	+ 6	+ 5	0	0
5432	+ 26	+ 31	− 18	0	− 36	− 15	+ 56	+ 3	− 6	− 17	− 4	− 12	+ 11	0	0
5431^2	− 53	− 10	+ 11	0	+ 24	+ 36	0	− 38	+ 3	− 12	+ 20	+ 12	− 11	0	0
53^3	+ 6	+ 6	− 3	0	+ 10	− 11	+ 4	− 3	+ 5	+ 6	− 3	− 6	+ 3	0	0
4^32	+ 2	− 6	0	0	+ 10	+ 4	− 12	+ 10	+ 2	− 12	0	+ 6	0	0	0
721^5	+ 34	+ 32	− 34	+ 7	+ 36	− 41	− 36	+ 41	− 18	+ 77	− 41	+ 12	− 59	+ 41	− 7
6321^3	−141	−132	+149	− 32	+ 56	− 32	− 48	+ 16	− 45	+ 92	− 19	+ 12	− 39	+ 11	0
542^21	− 16	− 47	+ 19	0	+ 24	+ 32	− 16	− 17	− 30	+ 37	− 1	0	− 5	0	0
4^23^2	+ 10	− 10	0	0	− 9	+ 30	− 10	− 11	0	+ 1	0	+ 3	0	0	0
4^31^2	− 11	+ 6	0	0	+ 4	+ 6	+ 16	− 15	− 16	+ 17	0	− 6	0	0	0
53^221	− 32	− 45	+ 19	0	− 24	+ 40	− 56	+ 5	+ 15	− 22	+ 13	+ 24	− 12	0	0
4^2321	− 28	+ 26	0	0	+ 24	− 38	0	+ 3	0	+ 21	0	− 12	0	0	0
43^31	− 10	+ 7	0	0	+ 8	− 36	+ 16	+ 11	− 5	− 4	0	0	0	0	0
532^3	− 4	− 25	+ 9	0	+ 40	− 6	− 16	+ 2	− 15	+ 17	− 5	+ 4	− 2	0	0
4^22^3	+ 6	+ 2	0	0	− 28	0	+ 16	− 10	0	+ 4	0	− 2	0	0	0
43^22^2	+ 1	+ 7	0	0	+ 12	− 6	− 8	+ 6	+ 15	− 7	0	0	0	0	0
3^42	0	0	0	0	− 10	+ 11	+ 2	0	− 5	0	0	0	0	0	0
5^21^4	+ 3	+ 2	− 1	0	+ 13	− 9	− 1	+ 5	0	+ 1	− 2	− 1	+ 1	0	0
5421^3	+ 28	+ 26	− 12	0	+ 8	− 44	− 8	+ 32	+ 15	− 14	− 8	0	+ 5	0	0
53^21^3	+ 17	+ 12	− 6	0	+ 4	− 22	+ 12	+ 8	− 15	+ 22	− 11	− 12	+ 6	0	0
4^231^3	+ 24	− 13	0	0	− 20	+ 2	− 16	+ 12	+ 15	− 16	0	+ 6	0	0	0
62^31^2	− 75	− 53	+ 71	− 16	+ 44	− 6	− 48	+ 6	− 15	+ 51	− 5	+ 18	− 36	+ 9	0
532^21^2	+ 35	+ 81	− 30	0	− 12	− 24	+ 56	− 4	0	+ 1	− 5	− 18	+ 9	0	0
$4^22^21^2$	+ 4	− 9	0	0	+ 18	− 6	− 16	+ 18	+ 15	− 23	0	+ 9	0	0	0
43^221^2	+ 28	− 21	0	0	− 36	+ 60	+ 12	− 24	− 15	+ 9	0	0	0	0	0
3^41^2	0	0	0	0	+ 1	+ 6	− 9	0	+ 5	0	0	0	0	0	0
52^41	+ 2	+ 21	− 7	0	− 26	+ 3	+ 4	− 2	+ 15	− 15	+ 5	0	0	0	0
432^31	− 8	− 7	0	0	− 8	+ 6	0	− 2	− 15	+ 5	0	0	0	0	0
3^32^21	0	0	0	0	+ 28	− 24	0	0	+ 5	0	0	0	0	0	0

	7^2	761	752	751^2	743	7421	741^3	73^21	732^2	7321^2	731^4	72^31	72^21^3	6^22	6^21^2
653	+ 14	− 8	− 11	− 9	− 23	+ 42	− 8	− 6	+ 6	+ 15	− 8	− 18	+ 8	+ 20	− 6
6521	+ 7	− 58	+ 2	+ 39	+ 37	− 27	− 5	− 11	− 5	+ 6	+ 4	+ 3	− 3	− 12	+ 27
651^3	− 21	+ 37	+ 20	− 25	− 3	− 8	+ 8	+ 3	+ 1	0	− 2	− 1	+ 1	+ 16	− 16
64^2	+ 7	− 4	− 28	+ 18	+ 12	− 20	+ 8	− 10	+ 22	0	− 4	− 8	+ 4	+ 10	− 3
6431	+ 7	− 10	+ 25	− 3	− 32	+ 29	− 17	+ 37	− 25	− 36	+ 16	+ 28	− 9	− 12	+ 3
642^2	− 21	− 12	+ 36	− 6	+ 24	− 4	− 4	− 18	− 20	+ 24	− 4	+ 4	− 2	− 6	+ 9
6421^2	− 14	+ 52	− 24	− 13	+ 2	− 12	+ 15	− 5	+ 12	+ 5	− 6	− 8	+ 4	+ 12	− 20
641^4	+ 14	− 22	− 4	+ 9	− 2	+ 6	− 5	+ 2	− 2	− 3	+ 2	+ 2	− 1	− 8	+ 8
63^22	− 21	+ 12	+ 15	+ 3	+ 18	− 21	+ 5	+ 6	− 21	− 3	+ 2	+ 15	− 5	− 6	− 3
63^21^2	− 7	+ 2	− 13	− 2	+ 25	− 11	+ 6	− 31	+ 15	+ 27	− 9	− 15	+ 5	0	+ 5
632^21	+ 35	− 1	− 29	− 2	− 35	+ 26	− 6	+ 2	+ 29	− 7	0	− 15	+ 5	0	+ 2
62^4	+ 7	0	− 4	+ 2	− 8	0	0	+ 8	+ 4	− 8	+ 2	0	0	+ 2	− 1
5^24	+ 7	− 28	+ 17	− 3	− 8	− 1	+ 9	+ 22	− 13	− 18	0	+ 9	0	− 14	+ 21
5^231	− 21	+ 23	− 6	− 2	+ 31	− 15	0	− 18	+ 7	+ 10	0	− 4	0	− 6	− 2
5^22^2	+ 14	− 7	− 6	+ 8	− 17	+ 15	− 9	− 4	+ 4	+ 4	0	− 2	0	+ 9	− 1
5^221^2	− 7	+ 23	+ 8	− 16	− 17	− 1	+ 9	+ 17	− 5	− 11	0	+ 4	0	0	− 7
54^21	− 21	+ 47	− 6	− 6	+ 4	+ 12	− 9	− 21	+ 4	+ 14	0	− 5	0	+ 18	− 26
5432	+ 7	+ 22	− 43	+ 18	+ 8	− 21	0	− 2	+ 17	+ 4	0	− 5	0	+ 12	− 17
5431^2	+ 35	− 58	+ 21	− 2	− 23	− 4	+ 9	+ 23	− 7	− 13	0	+ 5	0	− 12	+ 23
53^3	− 7	+ 4	+ 13	− 3	− 5	− 12	+ 6	0	+ 6	− 3	0	0	0	− 10	+ 3
4^32	− 7	+ 4	+ 20	− 14	− 12	+ 4	0	+ 10	− 6	0	0	0	0	− 14	+ 5
721^5	− 7	+ 1	+ 7	− 1	+ 7	− 8	+ 1	− 4	− 7	+ 9	− 1	+ 5	− 5	0	0
6321^3	+ 7	− 16	+ 23	+ 5	− 19	+ 6	− 5	+ 28	− 17	− 23	+ 8	+ 15	− 5	+ 4	− 3
542^21	− 14	+ 1	+ 19	− 24	+ 2	+ 1	+ 9	+ 10	− 5	− 12	0	+ 5	0	− 24	+ 13
4^23^2	+ 14	− 19	− 7	− 2	+ 4	+ 13	0	− 4	− 3	0	0	0	0	+ 9	+ 5
4^31^2	+ 14	− 19	− 7	+ 9	+ 4	− 3	0	− 1	+ 1	0	0	0	0	− 2	+ 5
53^221	+ 35	− 28	− 11	+ 11	− 11	+ 43	− 18	− 4	− 14	+ 7	0	0	0	+ 24	− 7
4^2321	− 14	+ 1	− 3	+ 12	+ 20	− 15	0	− 8	+ 7	0	0	0	0	+ 12	− 5
43^31	− 21	+ 26	+ 1	+ 2	− 3	− 7	0	+ 4	0	0	0	0	0	− 8	− 5
532^3	− 21	+ 2	+ 8	− 2	+ 17	− 9	+ 3	− 2	− 4	+ 2	0	0	0	− 4	+ 1
4^22^3	+ 14	0	− 20	+ 10	+ 4	0	0	− 4	+ 2	0	0	0	0	+ 10	− 5
43^22^2	− 7	+ 2	+ 22	− 9	− 17	+ 3	0	+ 2	0	0	0	0	0	− 12	+ 5
3^42	+ 7	− 4	− 7	0	+ 5	0	0	0	0	0	0	0	0	+ 4	0
5^21^4	+ 7	− 11	− 7	+ 7	+ 5	+ 2	− 3	− 5	+ 1	+ 3	0	− 1	0	− 4	+ 4
5421^3	+ 7	− 10	− 17	+ 24	+ 17	+ 4	− 12	− 20	+ 5	+ 14	0	− 5	0	+ 16	− 9
53^21^3	− 21	+ 23	+ 1	− 5	+ 3	− 6	+ 3	0	+ 2	− 1	0	0	0	− 4	− 2
4^231^3	− 21	+ 26	+ 1	− 8	− 3	+ 5	0	0	− 1	0	0	0	0	+ 2	− 5
62^31^2	− 21	+ 2	+ 6	− 3	+ 25	− 9	+ 3	− 16	− 8	+ 16	− 4	0	0	− 2	+ 1
532^21^2	− 14	+ 10	+ 4	− 5	+ 2	− 24	+ 9	+ 4	+ 8	− 4	0	0	0	− 6	+ 4
$4^22^21^2$	− 7	+ 2	+ 22	− 12	− 17	+ 3	0	+ 8	− 4	0	0	0	0	− 9	+ 5
43^221^2	+ 35	− 22	− 15	0	+ 7	+ 6	0	− 4	0	0	0	0	0	0	+ 5
3^41^2	+ 7	− 7	+ 3	0	− 1	0	0	0	0	0	0	0	0	+ 2	0
52^41	+ 14	− 1	− 4	+ 2	− 9	+ 9	− 3	0	0	0	0	0	0	+ 2	− 1
432^31	+ 7	− 2	− 12	+ 8	+ 9	− 3	0	0	0	0	0	0	0	+ 8	− 5
3^32^21	− 21	+ 9	+ 6	0	− 3	0	0	0	0	0	0	0	0	− 4	0

	5^221^2	54^21	5432	5431^2	53^3	4^32	721^5	6321^3	542^21	4^23^2	4^31^2	53^221	4^2321	43^31	532^3
653	+ 6	− 30	+ 9	− 3	+ 1	+ 4	0	+ 1	+ 9	− 3	+ 13	− 3	− 15	+ 5	− [illegible]
6521	− 8	+ 6	+ 10	+ 12	+ 6	+ 4	0	− 8	+ 8	− 8	− 8	− 19	0	+ 5	+
651^3	+ 7	+ 11	+ 2	− 13	− 3	0	0	+ 3	− 3	0	0	+ 7	0	0	−
64^2	− 16	− 2	− 20	+ 8	+ 2	+ 10	0	+ 8	+ 16	− 1	− 4	+ 8	− 8	0	−
6431	+ 22	+ 20	+ 23	− 8	0	− 12	0	− 10	− 25	+ 1	+ 1	+ 1	+ 5	− 1	+ 2
642^2	+ 16	− 6	− 4	− 4	+ 6	+ 2	0	− 4	0	− 3	+ 2	− 8	+ 4	0	+ 4
6421^2	− 15	− 6	− 2	+ 5	− 3	0	0	+ 11	+ 2	0	0	+ 4	0	0	− 2
641^4	0	0	0	0	0	0	0	− 3	0	0	0	0	0	0	0
63^22	− 15	0	− 9	+ 3	− 6	+ 2	0	+ 5	+ 3	+ 3	− 1	+ 6	− 3	0	0
63^21^2	− 6	− 1	− 6	0	+ 3	0	0	− 1	+ 6	0	0	− 3	0	0	0
632^21	+ 15	+ 1	+ 5	− 1	0	0	0	− 3	− 3	0	0	0	0	0	0
62^4	− 4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5^24	+ 11	+ 12	+ 9	+ 2	− 1	− 6	0	0	− 13	+ 1	− 3	− 3	+ 7	− 1	+ 4
5^231	− 6	+ 11	− 18	− 2	0	+ 2	0	0	+ 5	+ 2	− 3	+ 2	+ 4	− 2	− 1
5^22^2	− 1	− 3	+ 6	− 6	− 6	− 8	0	0	0	+ 8	+ 3	+ 6	− 3	0	− 2
5^221^2	$\boxed{+1}$	− 5	− 2	+ 5	+ 3	0	0	0	− 1	0	0	− 3	0	0	+ 1
54^21	− 5	$\boxed{-15}$	+ 2	− 2	0	+ 2	0	0	+ 5	− 1	+ 3	− 1	− 3	+ 1	
5432	− 2	+ 2	$\boxed{-7}$	+ 8	+ 6	+ 4	0	0	− 1	− 3	− 3	− 2	+ 1		
5431^2	+ 5	− 2	+ 8	$\boxed{-1}$	− 3	0	0	0	− 2	0	0	+ 1			
53^3	+ 3	0	+ 6	− 3	$\boxed{0}$	− 2	0	0	0	0	+ 1				
4^32	0	+ 2	+ 4	0	− 2	$\boxed{-2}$	0	0	0	+ 1					
721^5	0	0	0	0	0	0	$\boxed{+1}$								
6321^3	0	0	0	0	0	0		$\boxed{+1}$							
542^21	− 1	+ 5	− 1	− 2	0	0			$\boxed{+1}$						
4^23^2	0	− 1	− 3	0	0	+ 1									
4^31^2	0	+ 3	− 3	0	+ 1										
53^221	− 3	− 1	− 2	+ 1											
4^2321	0	− 3	+ 1												
43^31	0	+ 1													
532^3	+ 1														

	4^22^3	43^22^2	3^42	5^21^4	5421^3	53^21^3	4^231^3	62^31^2	532^21^2	$4^22^21^2$	43^221^2	3^41^2	52^41	432^31	3^32^21
653	0	+ 3	− 1	− 4	+ 1	+ 8	− 7	− 3	− 9	+ 3	+ 6	− 2	+ 3	− 3	+ 1
6521	0	− 1	0	+ 8	− 10	− 4	+ 5	+ 3	+ 11	− 1	− 3	0	− 3	+ 1	
651^3	0	0	0	− 4	+ 4	+ 2	0	− 1	− 4	0	0	0	+ 1		
64^2	− 4	+ 8	− 2	+ 6	− 8	− 4	+ 4	− 4	+ 4	+ 2	− 4	+ 1			
6431	+ 4	− 2	0	− 8	+ 10	0	− 1	+ 4	− 1	− 2	+ 1				
642^2	− 2	0	0	− 4	0	+ 4	− 2	+ 2	− 2	+ 1					
6421^2	0	0	0	+ 4	− 1	− 2	0	− 4	+ 1						
641^4	0	0	0	0	0	0	0	+ 1							
63^22	0	0	0	+ 4	− 1	− 2	+ 1								
63^21^2	0	0	0	+ 2	− 2	+ 1									
632^21	0	0	0	− 4	+ 1										
62^4	0	0	0	+ 1											
5^24	+ 2	− 4	+ 1												
5^231	− 2	+ 1													
5^22^2	+ 1														

	14	13.1	12.2	12.1^2	11.3	11.21	11.1^3	10.4	10.31	10.2^2	10.21^2	101^4	95	941	932
42^5	+ 14	− 14	+ 10	+ 2	− 14	+ 4	− 2	+ 14	0	− 4	+ 2	0	− 14	0	+ 4
3^22^4	+ 35	− 35	+ 13	+ 11	− 2	− 11	0	− 5	+ 7	+ 2	0	0	+ 10	− 5	− 2
631^5	− 84	+ 19	+ 36	− 19	+ 18	− 22	+ 8	+ 24	− 27	− 16	+ 31	− 8	+ 30	− 34	− 36
541^5	− 84	+ 19	+ 36	− 19	+ 51	− 55	+ 19	+ 24	− 30	− 16	+ 34	− 9	− 15	+ 11	+ 3
62^21^4	−210	+ 80	+ 18	− 20	+ 45	− 54	+ 20	+ 60	− 75	− 28	+ 74	− 20	+ 75	− 95	− 72
5321^4	−420	+160	+156	−100	+ 57	−107	+ 45	+120	−147	− 76	+167	− 45	− 75	+ 35	+ 57
4^221^4	−210	+ 80	+ 78	− 50	+111	−136	+ 50	− 60	+ 24	+ 22	− 14	0	− 15	+ 25	− 9
43^21^4	−210	+ 80	+138	− 80	+ 12	− 53	+ 25	− 60	+ 48	− 8	− 7	0	− 15	+ 25	− 6
52^31^3	−280	+150	− 8	− 30	+ 71	− 76	+ 30	+ 80	−111	− 32	+106	− 30	− 50	+ 10	+ 60
432^21^3	−840	+450	+216	−210	+ 81	−204	+100	−120	+ 69	+ 64	− 46	0	+ 30	+ 30	− 36
3^321^3	−280	+150	+152	−110	−127	+ 50	0	0	+ 47	− 32	0	0	+ 40	− 30	+ 11
42^41^2	−210	+145	− 42	− 25	+ 78	− 59	+ 25	− 50	+ 7	+ 32	− 16	0	+ 30	− 5	− 18
$3^22^31^2$	−420	+290	+ 36	−110	−108	+ 70	0	+ 80	− 7	− 16	0	0	− 30	− 10	+ 9
32^51	− 84	+ 71	− 36	− 11	+ 18	+ 9	0	− 4	− 7	0	0	0	− 6	+ 5	0
2^7	− 2	+ 2	− 2	0	+ 2	0	0	− 2	0	0	0	0	+ 2	0	0
71^7	+ 14	− 1	− 2	+ 1	− 3	+ 3	− 1	− 4	+ 4	+ 2	− 4	+ 1	− 5	+ 5	+ 5
621^6	+ 98	− 20	− 14	+ 8	− 21	+ 23	− 8	− 28	+ 31	+ 14	− 31	+ 8	− 35	+ 39	+ 35
531^6	+ 98	− 20	− 38	+ 20	− 21	+ 25	− 9	− 28	+ 31	+ 18	− 35	+ 9	+ 10	− 6	− 4
4^21^6	+ 49	− 10	− 19	+ 10	− 27	+ 29	− 10	+ 6	− 3	− 1	+ 1	0	+ 5	− 5	+ 1
52^21^5	+294	− 99	− 30	+ 27	− 63	+ 74	− 27	− 84	+102	+ 40	−101	+ 27	+ 30	− 6	− 33
4321^5	+588	−198	−204	+126	− 93	+149	− 60	+ 72	− 39	− 16	+ 15	0	+ 15	− 27	+ 9
3^31^5	+ 98	− 33	− 58	+ 33	+ 23	− 8	0	+ 12	− 15	+ 8	0	0	− 5	+ 3	− 1
42^31^4	+490	−230	− 10	+ 50	−116	+130	− 50	+ 60	− 14	− 40	+ 20	0	− 25	+ 5	+ 12
$3^22^21^4$	+735	−345	−195	+165	+156	− 65	0	− 30	− 21	+ 20	0	0	− 15	+ 15	− 6
32^41^3	+490	−295	+ 50	+ 55	+ 16	− 30	0	− 30	+ 14	0	0	0	+ 20	− 5	0
2^61^2	+ 49	− 36	+ 25	0	− 16	0	0	+ 9	0	0	0	0	− 4	0	0
61^8	− 14	+ 1	+ 2	− 1	+ 3	− 3	+ 1	+ 4	− 4	− 2	+ 4	− 1	+ 5	− 5	− 5
521^7	−112	+ 21	+ 16	− 9	+ 24	− 26	+ 9	+ 32	− 35	− 16	+ 35	− 9	− 5	+ 1	+ 5
431^7	−112	+ 21	+ 40	− 21	+ 24	− 28	+ 10	− 8	+ 5	0	− 1	0	− 5	+ 5	− 1
42^21^6	−392	+119	+ 44	− 35	+ 84	− 97	+ 35	− 28	+ 7	+ 16	− 8	0	+ 5	− 1	− 2
3^221^6	−392	+119	+128	− 77	− 48	+ 17	0	− 8	+ 14	− 8	0	0	+ 5	− 3	+ 1
32^31^5	−784	+329	+ 40	− 77	− 52	+ 27	0	+ 24	− 7	0	0	0	− 5	+ 1	
2^51^4	−196	+105	− 50	0	+ 20	0	0	− 6	0	0	0	0	+ 1		
51^9	+ 14	− 1	− 2	+ 1	− 3	+ 3	− 1	− 4	+ 4	+ 2	− 4	+ 1			
421^8	+126	− 22	− 18	+ 10	− 27	+ 29	− 10	+ 4	− 1	− 2	+ 1				
3^21^8	+ 63	− 11	− 21	+ 11	+ 3	− 1	0	+ 2	− 2	+ 1					
32^21^7	+504	−140	− 60	+ 44	+ 24	− 9	0	− 4	+ 1						
2^41^6	+294	−112	+ 35	0	− 8	0	0	+ 1							
41^{10}	− 14	+ 1	+ 2	− 1	+ 3	− 3	+ 1								
321^9	−140	+ 28	+ 20	− 11	− 3	+ 1									
2^31^8	−210	+ 54	− 10	0	+ 1										
31^{11}	+ 14	− 1	− 2	+ 1											
2^21^{10}	+ 77	− 12	+ 1												
21^{12}	− 14	+ 1													
1^{14}	+ 1														

	7^2	761	752	751^2	743	7421	741^3	73^21	732^2	7321^2	731^4	72^31	72^21^3	6^22	6^21^2
42^5	-7	0	$+4$	-2	0	0	0	0	0	0	0	0	0	-2	$+1$
3^22^4	$+7$	-1	-2	0	0	0	0	0	0	0	0	0	0	$+1$	
631^5	-7	$+7$	-3	-2	$+7$	-4	$+2$	-7	$+3$	$+6$	-2	-3	$+1$		
541^5	-7	$+7$	$+7$	-7	-5	-2	$+3$	$+5$	-1	-3	0	$+1$			
62^21^4	$+7$	-1	-2	$+1$	-7	$+3$	-1	$+4$	$+2$	-4	$+1$				
5321^4	$+14$	-3	-4	$+3$	-2	$+3$	-3	-1	-2	$+1$					
4^221^4	$+7$	-4	-7	$+4$	$+5$	-1	0	-2	$+1$						
43^21^4	$+7$	-7	$+3$	$+2$	-1	-2	0	$+1$							
52^31^3	-7	$+1$	$+2$	-1	$+3$	-3	$+1$								
432^21^3	-21	$+9$	$+6$	-4	-3	$+1$									
3^321^3	-7	$+5$	-3	0	$+1$										
42^41^2	$+7$	-1	-2	$+1$											
$3^22^31^2$	$+14$	-5	$+1$												
32^51	-7	$+1$													
2^7	$+1$														

Lectures on the Theory of Reciprocants.

BY PROFESSOR SYLVESTER, F. R. S., *Savilian Professor of Geometry in the University of Oxford.*

[Reported by JAMES HAMMOND, M. A.]

LECTURE XXV.

In a letter to me dated June 14th, 1886, M. Halphen calls forms which are persistent under the substitution $\frac{1}{x}$, $\frac{y}{x}$, *Invariants d'homologie.* He uses the letters

$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots\ a_n$$

to denote y and its successive modified derivatives with respect to x; and, supposing them to become

$$A_0,\ A_1,\ A_2,\ A_3,\ \ldots\ A_n$$

in consequence of the substitution $\frac{1}{x}$, $\frac{y}{x}$, gives, in the briefest possible manner, two very ingenious proofs of the formula

$$A_n = (-)^n x^{2n-1}\left\{a_n + \frac{n-2}{1.x}\,a_{n-1} + \frac{(n-2)(n-3)}{1.2.x^2}\,a_{n-2} + \ldots\right\},$$

from which he deduces the theorem that the substitution in question changes any homogeneous and isobaric function f, of degree i and weight ω in

$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots\ a_n,$$

into

$$F = (-)^\omega x^{2\omega - i} e^{\frac{\Theta}{x}} f,$$

where Θ is the partial differential operator

$$-a_0\partial_{a_1} + a_2\partial_{a_3} + 2a_3\partial_{a_4} + \ldots + (n-2)\,a_{n-1}\partial_{a_n}.$$

I give the two proofs mentioned above in M. Halphen's own words, adding occasional footnotes, and making slight changes in the literation of his formulae when it seems desirable to do so.

Soient $\qquad X = \frac{1}{x},\quad Y = \frac{y}{x}.$

Par une formule connue (Schlömilch, Compendium II.)

$$\frac{d^n y}{dX^n} = (-1)^n x^{n+1} \frac{d^n}{dx^n}(x^{n-1}y)^*$$

et puisque $Y = Xy$, il en resulte

$$\frac{d^n Y}{dX^n} = X\frac{d^n y}{dX^n} + n\frac{d^{n-1}y}{dX^{n-1}} = (-1)^n x^n \left\{\frac{d^n}{dx^n}(x^{n-1}y) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-2}y)\right\}$$

$$= (-1)^n x^{2n-1}\left\{y_n + \frac{n(n-2)}{1.x}y_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.2.x^2}y_{n-2} + \dots\right\}.^\dagger$$

Si l'on pose $\dfrac{d^n Y}{dX^n} = n!\,A_n,\ y_n = n!\,a_n$

il vient

$$\text{(I)}\qquad A_n = (-1)^n x^{2n-1}\left\{a_n + \frac{n-2}{1.x}a_{n-1} + \frac{(n-2)(n-3)}{1.2.x^2}a_{n-2} + \dots\right\}.$$

Soit $$\Theta f = \Sigma\,(n-2)\,a_{n-1}\frac{\partial f}{\partial a_n}\ ^\ddagger$$

on aura

$$\Theta a_n = (n-2)\,a_{n-1},$$
$$\Theta^2 a_n = (n-2)(n-3)\,a_{n-2},$$
$$\dots\dots\dots\dots\dots\dots$$
$$A_n = (-1)^n x^{2n-1}\left\{a_n + \frac{1}{1.x}\Theta a_n + \frac{1}{1.2.x^2}\Theta^2 a_n + \dots\right\}.$$

---

* An easy inductive proof of this may be obtained as follows:

Since $\dfrac{d}{dX} = -x^2\dfrac{d}{dx}$ we have $\dfrac{d^{\kappa+1}y}{dX^{\kappa+1}} = -x^2\dfrac{d}{dx}\left(\dfrac{d^\kappa y}{dX^\kappa}\right)$.

Hence, assuming the truth of the formula when $n = \kappa$, we find

$$\frac{d^{\kappa+1}y}{dX^{\kappa+1}} = (-)^{\kappa+1}x^2\frac{d}{dx}\left\{x^{\kappa+1}\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}$$
$$= (-)^{\kappa+1}x^2\left\{x^{\kappa+1}\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^{\kappa-1}y) + (\kappa+1)\,x^\kappa\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}$$
$$= (-)^{\kappa+1}x^{\kappa+2}\left\{x\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^{\kappa-1}y) + (\kappa+1)\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}$$
$$= (-)^{\kappa+1}x^{\kappa+2}\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^\kappa y).$$

Thus, if the formula is true for $n = \kappa$, it will be equally so when $n = \kappa + 1$. But it is obviously true when $n = 1$ $\left(\text{when it becomes } \dfrac{dy}{dX} = -x^2\dfrac{dy}{dx}\right)$, and therefore holds universally.

† For, expanding by Leibnitz's Theorem,

$$\frac{d^n}{dx^n}(x^{n-1}y) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-2}y) = x^{n-1}y_n + n(n-1)\,x^{n-2}y_{n-1} + \frac{n(n-1)}{1.2}(n-1)(n-2)\,x^{n-3}y_{n-2} + \dots$$
$$- n\{x^{n-2}y_{n-1} + (n-1)(n-2)\,x^{n-3}y_{n-2} + \dots\}$$
$$= x^{n-1}y_n + n(n-2)\,x^{n-2}y_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.2}x^{n-3}y_{n-2} + \dots$$

‡ The summation extending to all positive integral values of n, from 1 to ∞, so that

$$\Theta = -a_0\partial_{a_1} + a_2\partial_{a_3} + 2a_3\partial_{a_4} + 3a_4\partial_{a_5} + \dots$$

Remembering that Halphen's $a_0, a_1, a_2, a_3, \dots$ have the same meaning as our $y, t, a, b, \dots$, this operator is $-y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \dots$ identical with the Θ used in previous lectures.

Par conséquent, pour une fonction contenant $a_0, a_1, a_2, \ldots.$, de degré i et de poids ω, à chaque terme, on aura

$$F=(-1)^{\omega}x^{2\omega-i}\left\{f+\frac{1}{1.x}\Theta f+\frac{1}{1.2.x^2}\Theta^2 f+\ldots.\right\}.^*$$

C. Q. F. D.

Autre Demonstration de la Formule (I).†

Si l'on change X et x en $X+H$ et $x+h$, on a

$$h=-\frac{H}{X(X+H)}.$$

Maintenant la formule

$$y=a_0+ha_1+h^2a_2+\ldots.+h^na_n+\ldots.$$

écrite *symboliquement* ‡

$$y=\frac{1}{1-ah}$$

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* We may show without much difficulty that, when $\Theta_1, \Theta_2, \Theta_3, \ldots.$ are each of them equivalent to Θ, but Θ_1 acts on u only, Θ_2 on v, Θ_3 on w, and so on, $\Theta uvw\ldots.=(\Theta_1+\Theta_2+\Theta_3+\ldots.)uvw\ldots.$ From this it can be deduced that $\Theta^{\kappa}uvw\ldots.=(\Theta_1+\Theta_2+\Theta_3+\ldots.)^{\kappa}uvw\ldots.$, when κ is any positive integer. Now let the number of the functions $u, v, w, \ldots.$ be i, and suppose that

$$u=a_n,\ v=a_p,\ w=a_q,\ \ldots.;$$

suppose, also, that the weight $n+p+q+\ldots.=\omega$. Then

$$A_nA_pA_q\ldots.=(-)^{\omega}x^{2\omega-i}\left(e^{\frac{\Theta}{x}}a_n\right)\left(e^{\frac{\Theta}{x}}a_p\right)\left(e^{\frac{\Theta}{x}}a_q\right)\ldots.=(-)^{\omega}x^{2\omega-i}e^{\frac{1}{x}(\Theta_1+\Theta_2+\Theta_3+\ldots.)}a_na_pa_q\ldots.$$

$$=(-)^{\omega}x^{2\omega-i}e^{\frac{\Theta}{x}}a_na_pa_q\ldots.$$

(for by what precedes $\Theta_1+\Theta_2+\Theta_3+\ldots.$ may be replaced by Θ). Taking $a_na_pa_q\ldots.$ and $A_nA_pA_q\ldots.$ to be corresponding terms of f and F, we see at once that

$$F=(-)^{\omega}x^{2\omega-i}e^{\frac{\Theta}{x}}f.$$

† If x becomes $x+h$ in consequence of the augmentation of X by an arbitrary quantity H, the increment of x will not be a constant, but will depend on X as well as on H. The value of h may be found at once by eliminating x between $X=\frac{1}{x}$ and $X+H=\frac{1}{x+h}$, when we obtain $X+H=\frac{X}{1+hX}$, and consequently $h=-\frac{H}{X(X+H)}$.

This increase of X also changes y and Y (functions of x and X, whose original values were a_0 and A_0 before the augmentation of X took place) into

$$y=a_0+ha_1+h^2a_2+\ldots.+h^na_n+\ldots.$$

and into

$$Y=A_0+HA_1+H^2A_2+\ldots.+H^nA_n+\ldots.$$

These altered values of y and Y are the ones used in this second proof; the other letters retain their original signification.

‡ The word *symboliquement* indicates, whenever it is used, that powers of a are to be replaced by suffixes of corresponding value. *E. g.* in the final result $A_n=(-)^nx^{2n-1}\left(a^n+\frac{n-2}{x}a^{n-1}+\ldots.\right)$ is to be replaced by $A_n=(-)^nx^{2n-1}\left(a_n+\frac{n-2}{x}a_{n-1}+\ldots.\right)$.

In our notation the final result is $A_{n+2}=(-)^nx^{2n+3}\left(a, b, c, d, \ldots.\between \frac{1}{x}, 1\right)^n$.

devient
$$y=\frac{X(X+H)}{X^2+H(X+a)}.$$
D'ailleurs
$$Y=(X+H)y;$$
donc *symboliquement*

(II)
$$Y=\frac{X(X+H)^2}{X^2+H(X+a)}.$$

Si l'on developpe le second membre (II) suivant les puissances ascendants de H, le coefficient de H^n est A_n. Or ce developpement est

$$Y=X\left\{1+\left(1-\frac{a}{X}\right)\frac{H}{X}+\left(\frac{a}{X}\right)^2\left(\frac{H}{X}\right)^2 + \ldots + (-1)^n\left(\frac{H}{X}\right)^n\left(1+\frac{a}{X}\right)^{n-2}\left(\frac{a}{X}\right)^2+\ldots\right\}$$

donc *symboliquement*

$$A_n=(-1)^n\frac{1}{X^{n+1}}\left(1+\frac{a}{X}\right)^{n-2}a^2=(-1)^n x^{2n-1}\left(a+\frac{1}{x}\right)^{n-2}a^2$$

ce qui est justement la formule (I).

We may regard the coefficients $a, b, c, \ldots$ of the ordinary binary Quantic in u, v,
$$(a, b, c, \ldots \between u, v)^n,$$
as the successive modified derivatives, beginning with the second, of a new variable y with respect to another new variable x.

Any invariant I of this Quantic will then retain its form unaltered, or at most merely acquire an extraneous factor, if

1° leaving x, y, v unaltered we change u into $u+\lambda v$,

2° " u, v " " " x, y " $\frac{x}{1+hx}$, $\frac{y}{1+hx}$,

3° " u, v " " " x, y " $\frac{1}{x}$, $\frac{y}{x}$,

where λ and h are arbitrary constants.

For we have seen that these three substitutions will severally convert any homogeneous and isobaric function F, of degree i and weight w in the letters $a, b, c, \ldots$, into

$$e^{\lambda\Omega}F,\ (1+hx)^{\nu}e^{\frac{h\Omega}{1+hx}}F,\ \text{and}\ (-)^w x^{\nu}e^{\frac{\Omega}{x}}F,$$

where, in each case, $\Omega=a\partial_b+2b\partial_c+3c\partial_d+\ldots$, and $\nu=3i+2w$. From our point of view an invariant is defined as a homogeneous and isobaric solution of the equation
$$\Omega I=0.$$
Hence the above substitutions convert the invariant I into
$$I,\ (1+hx)^{\nu}I,\ \text{and}\ (-)^w x^{\nu}I,\ \text{respectively.}$$

An *absolute invariant* with respect to any substitution is one which, disregarding its sign, remains unchanged in absolute value by that substitution. Thus, any invariant for which

$$\nu = 3i + 2w = 0$$

is an absolute invariant with respect to each of the three substitutions here considered.

An invariant is of odd or even character with respect to any substitution according as its sign is or is not changed by that substitution. Thus, invariants are of odd or even character with respect to the substitution $\frac{1}{x}$, $\frac{y}{x}$ according as their *weights* are odd or even.

This corresponds to the theorem that the character (with respect to the interchange of x and y) of a pure reciprocant is odd or even according as its degree is odd or even (vide *American Journal of Mathematics*, Vol. VIII, p. 251).

From any two invariants for which ν has the same value we can form an absolute invariant (*i. e.* one for which $\nu = 0$) by taking their ratio, and then by differentiating the absolute invariant thus formed obtain another invariant.

Suppose I_1 to be an invariant of degree i_1 and weight w_1,
I_2 " " " " " " i_2 " " w_2,

and let $3i_1 + 2w_1 = \nu_1$, $3i_2 + 2w_2 = \nu_2$;

then the ν for $I_1^{\nu_2}$ is the same as that for $I_2^{\nu_1}$, and consequently $I_1^{\nu_2} I_2^{-\nu_1}$ is an absolute invariant.

We proceed to show that $\frac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1})$ is an invariant, though not an absolute one.

Using accents to denote differential derivation with respect to x, we have

$$\frac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1}) = I_1^{\nu_2 - 1} I_2^{-\nu_1 - 1}(\nu_2 I_1' I_2 - \nu_1 I_1 I_2') .$$

If, then, we can prove that $\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$ is an invariant, it will follow that $\frac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1})$ will be one also, and the proposition will be established. It may be very easily shown that this is the case by using Cayley's generators P and Q. For (see *American Journal of Mathematics*, Vol. VIII, p. 221), I being any invariant of degree i and weight w, PI and QI are also invariants where

$$P = a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \dots) - ib,$$

and

$$Q = a(c\partial_b + 2d\partial_c + 3e\partial_d + \dots) - 2wb.$$

Hence $$(3P + Q)\, I \text{ is an invariant.}$$

Now, since $$3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots = \frac{d}{dx},$$

and $$3i + 2w = \nu,$$

$$(3P + Q)\, I = a\,(3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots)\, I - (3i + 2w)\, bI = aI' - \nu bI.$$

Consequently $$aI_1' - \nu_1 bI_1 \text{ and } aI_2' - \nu_2 bI_2$$

are both of them invariants. Hence the combination

$$\nu_2 I_2 (aI_1' - \nu_1 bI_1) - \nu_1 I_1 (aI_2' - \nu_2 bI_2) = a\,(\nu_2 I_1' I_2 - \nu_1 I_1 I_2')$$

is also an invariant; *i. e.*

$$\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$$

is one; which is the theorem to be demonstrated.

The invariant $aI' - \nu bI$, which we generated from I, is of degree $i + 1$ and weight $w + 1$; its ν is therefore the original ν increased by 5 units, three for the unit increase in the degree and two for the unit increase in the weight. Hence, on repeating the process of generation, we obtain the invariant

$$\left\{a\frac{d}{dx} - (\nu + 5)\, b\right\}(aI' - \nu bI) = a^2 I'' - 2\,(\nu + 1)\, abI' - 4\nu acI + \nu\,(\nu + 5)\, b^2 I.$$

By adding on the invariant $\nu\,(\nu + 5)(ac - b^2)\, I$ and dividing the sum by a, the above invariant is reduced to

$$aI'' - 2\,(\nu + 1)\, bI' + \nu\,(\nu + 1)\, cI,$$

which is an invariant of lower degree by unity than the unreduced form.

The results obtained above may be compared with the corresponding ones in the theory of reciprocants.

Thus to the invariants	correspond the reciprocants
I (deg. i, wt. w),	R (deg. i, wt. w),
$aI' - \nu bI$,	$aR' - \mu bR$,
$\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$,	$\mu_2 R_1' R_2 - \mu_1 R_1 R_2'$,
$aI'' - 2\,(\nu + 1)\, bI' + \nu\,(\nu + 1)\, cI$,	$5aR'' - 5\,(2\mu + 1)\, bR' + 4\mu\,(\mu - 1)\, cR$,
where $\nu = 3i + 2w$,	where $\mu = 3i + w$.

Defining a *plenarily absolute* form to be one whose degree and weight are both zero ($i = 0$, $w = 0$), the theorem I shall now prove may be stated as follows:

By differentiating a plenarily absolute principiant we obtain another principiant.

Let P be any principiant of degree i and weight w. Then, by what precedes, since P is both an invariant and a reciprocant,

$$a\frac{dP}{dx} - \nu bP \text{ is an invariant,}$$

and

$$a\frac{dP}{dx} - \mu bP \text{ is a reciprocant.}$$

Hence, when $\nu = 0$ (*i. e.* when $3i + 2w = 0$),

$$\frac{dP}{dx} \text{ is an invariant,}$$

and when $\mu = 0$ (*i. e.* when $3i + w = 0$),

$$\frac{dP}{dx} \text{ is a reciprocant.}$$

When both $\mu = 0$ and $\nu = 0$ (which happens when $i = 0$, $w = 0$),

$$\frac{dP}{dx} \text{ is both a reciprocant and an invariant;}$$

i. e.

$$\frac{dP}{dx} \text{ is a principiant.}$$

LECTURE XXVI.

In the theory of Invariants the annihilator Ω has two independent reversors any linear combination of which will also be a reversor. To each of these reversors there corresponds a generator for invariants. Thus Cayley's two generators

$$a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots\ldots) - ib,$$
$$a(c\partial_b + 2d\partial_c + 3e\partial_d + \ldots\ldots) - 2wb,$$

correspond to the two reversors

$$b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots\ldots,$$
$$c\partial_b + 2d\partial_c + 3e\partial_d + \ldots\ldots$$

The only linear combination of these which does not increase the extent j as well as the weight of the operand is

$$O = jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \ldots\ldots$$

It is convenient to take this for one of our reversors, and for the other

$$\frac{d}{dx} = 3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots\ldots,$$

which is a reversor to V, the annihilator for reciprocants, as well as to Ω, the annihilator for invariants.

We saw in Lecture XI (*American Journal of Mathematics*, Vol. IX, p. 1) that when F is any homogeneous and isobaric function of degree i and weight w in the $j+1$ letters $a, b, c, \ldots.$

$$(\Omega O - O\Omega)F = (ij - 2w)F.$$

The method employed in proving this can also be applied to show that

$$\left(\Omega \frac{d}{dx} - \frac{d}{dx}\Omega\right)F = \nu F,$$

where $\nu = 3i + 2w$.

Corresponding to the reversors O and $\frac{d}{dx}$ we have the two generators for invariants

$$a\frac{d}{dx} - \nu b \text{ and } aO - (ij - 2w)b,$$

which are linear combinations of Cayley's generators.

Thus, if I be any invariant,

$$\left(a\frac{d}{dx} - \nu b\right)I \text{ and } \{aO - (ij - 2w)b\}I$$

are also invariants.

The operator $\frac{d}{dx}$ has, but O has not, analogous properties in the theory of Reciprocants; viz. $\frac{d}{dx}$ is a reversor to V and $a\frac{d}{dx} - \mu b$ is a generator for reciprocants. Thus, we have shown in previous lectures that

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2\mu aF,$$

where F is any homogeneous and isobaric function, and $\mu = 3i + w$, and that if R is any pure reciprocant $\left(a\frac{d}{dx} - \mu b\right)R$ is one also.

Now, Mr. Hammond has found that if

$$W = \frac{b}{a}\partial_a + \frac{2ac - b^2}{a^2}\partial_b + \frac{3a^2d - 3abc + b^3}{a^3}\partial_c + \ldots,$$

W is a reversor to V, and $a^2W - ib$ is a generator for pure reciprocants. In fact we have

$$\begin{aligned} VW - WV = {} & V\left(\frac{b}{a}\right)\partial_a \\ & + \left\{V\left(\frac{2ac - b^2}{a^2}\right) - W(2a^2)\right\}\partial_b \\ & + \left\{V\left(\frac{3a^2d - 3abc + b^3}{a^3}\right) - W(5ab)\right\}\partial_c \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

But, since

$$V\left(\frac{b}{a}\right) \qquad = 2a,$$

$$V\left(\frac{2ac-b^2}{a^2}\right) \qquad = 10b - 4b = 6b,$$

$$V\left(\frac{3a^2d - 3abc + b^3}{a^3}\right) = \left(18c + 9\,\frac{b^2}{a}\right) - \left(15\,\frac{b^2}{a} + 6c\right) + 6\,\frac{b^2}{a} = 12c,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and

$$W(2a^2) = 4b,$$

$$W(5ab) = 5\,\frac{b^2}{a} + 5\left(\frac{2ac - b^2}{a}\right) = 10c,$$

$$\cdots\cdots\cdots\cdots$$

it follows that

$$VW - WV = 2a\partial_a + 2b\partial_b + 2c\partial_c + \ldots\ldots = 2i.$$

Thus W is a reversor to V. Moreover, $a^2W - ib$ acting on any pure reciprocant generates another.

Let R be a pure reciprocant of degree i; then, by what precedes,

$$(VW - WV)\,R = 2iR.$$

But, since R is a pure reciprocant, $VR = 0$, and consequently $VWR = 2iR$.

Now, $\quad V(a^2W - ib)\,R = a^2VWR - iRVb = a^2.\,2iR - iR.\,2a^2 = 0.$

Hence $\quad (a^2W - ib)\,R$

is a pure reciprocant; *i. e.* $\quad a^2W - ib$

is a generator for pure reciprocants.

Mr. Hammond shows that W is a reversor to V in the following manner:

Let

$$u = a_0 + a_1e^\theta + a_2e^{2\theta} + a_3e^{3\theta} + \ldots\ldots,$$

$$\phi(u) = A_0 + A_1e^\theta + A_2e^{2\theta} + A_3e^{3\theta} + \ldots\ldots,$$

$$\psi(u) = A_0' + A_1'e^\theta + A_2'e^{2\theta} + A_3'e^{3\theta} + \ldots\ldots,$$

and consider the operators

$$P = \lambda A_0\partial_{a_n} + (\lambda + \mu)\,A_1\partial_{a_{n+1}} + (\lambda + 2\mu)\,A_2\partial_{a_{n+2}} + \ldots\ldots,$$

$$Q = \lambda' A_0'\partial_{a_{n'}} + (\lambda' + \mu')\,A_1'\partial_{a_{n'+1}} + (\lambda' + 2\mu')\,A_2'\partial_{a_{n'+2}} + \ldots\ldots$$

Regarding e^θ as an operative symbol defined by the equation

$$e^{\kappa\theta}[\partial_{a_0}] = \partial_{a_\kappa},$$

we may write

$$P = \{\lambda A_0e^{n\theta} + (\lambda + \mu)\,A_1e^{(n+1)\theta} + (\lambda + 2\mu)\,A_2e^{(n+2)\theta} + \ldots\ldots\}\,[\partial_{a_0}]$$

$$= e^{n\theta}\lambda\,(A_0 + A_1e^\theta + A_2e^{2\theta} + \ldots\ldots)[\partial_{a_0}]$$

$$+ e^{n\theta}\mu\,(A_1e^\theta + 2A_2e^{2\theta} + \ldots\ldots)[\partial_{a_0}]$$

$$= e^{n\theta}\left(\lambda + \mu\,\frac{d}{d\theta}\right)\phi(u)[\partial_{a_0}].$$

Similarly,

$$Q = e^{n'\theta}\left(\lambda' + \mu' \frac{d}{d\theta}\right)\psi(u)[\partial_{a_0}].$$

Now,

$$PQ - QP = \left\{Pe^{n'\theta}\left(\lambda' + \mu' \frac{d}{d\theta}\right)\psi(u) - Qe^{n\theta}\left(\lambda + \mu \frac{d}{d\theta}\right)\phi(u)\right\}[\partial_{a_0}]$$

$$= \left\{e^{n'\theta}\left(\lambda' + \mu' \frac{d}{d\theta}\right)P\psi(u) - e^{n\theta}\left(\lambda + \mu \frac{d}{d\theta}\right)Q\phi(u)\right\}[\partial_{a_0}].$$

For $$Q\phi(u) = QA_0 + e^{\theta}QA_1 + e^{2\theta}QA_2 + \ldots ;$$

so that $$e^{n\theta}\frac{d}{d\theta}Q\phi(u) = e^{n\theta}(e^{\theta}QA_1 + 2e^{2\theta}QA_2 + \ldots)$$

and $$e^{n\theta}\frac{d}{d\theta}\phi(u) = e^{n\theta}(e^{\theta}A_1 + 2e^{2\theta}A_2 + \ldots);$$

so that $$Qe^{n\theta}\frac{d}{d\theta}\phi(u) = e^{n\theta}(e^{\theta}QA_1 + 2e^{2\theta}QA_2 + \ldots)$$

$$= e^{n\theta}\frac{d}{d\theta}Q\phi(u).$$

Similarly, $$Pe^{n'\theta}\frac{d}{d\theta}\psi(u) = e^{n'\theta}\frac{d}{d\theta}P\psi(u).$$

Moreover,

$$P\psi(u) = \psi'(u)Pu = \psi'(u)P(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \ldots)$$

$$= \psi'(u)\{e^{n\theta}\lambda A_0 + e^{(n+1)\theta}(\lambda + \mu)A_1 + e^{(n+2)\theta}(\lambda + 2\mu)A_2 + \ldots\}$$

$$= e^{n\theta}\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u).$$

Similarly, $$Q\phi(u) = e^{n'\theta}\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u).$$

Hence

$$PQ - QP = \left\{e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)e^{n\theta}\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right.$$

$$\left. - e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)e^{n'\theta}\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)\right\}[\partial_{a_0}]$$

$$= e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right.$$

$$\left. - \left(\lambda + \mu n' + \mu\frac{d}{d\theta}\right)\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)\right\}[\partial_{a_0}].$$

If in this we write

$$\phi = \frac{u^2}{2}, \quad \lambda = 4, \ \mu = 1, \ n = 1,$$

$$\psi = \log u, \ \lambda' = 0, \ \mu' = 1, \ n' = -1,$$

we have

$$PQ-QP=\left\{\left(1+\frac{d}{d\theta}\right)u^{-1}\left(4+\frac{d}{d\theta}\right)\frac{u^2}{2}-\left(3+\frac{d}{d\theta}\right)u\frac{d}{d\theta}\log u\right\}[\partial_{a_0}]$$
$$=\left\{\left(1+\frac{d}{d\theta}\right)\left(2u+\frac{du}{d\theta}\right)-\left(3+\frac{d}{d\theta}\right)\frac{du}{d\theta}\right\}[\partial_{a_0}]$$
$$=\left\{\left(1+\frac{d}{d\theta}\right)\left(2+\frac{d}{d\theta}\right)-\left(3+\frac{d}{d\theta}\right)\frac{d}{d\theta}\right\}u\,[\partial_{a_0}]$$
$$=2u\,[\partial_{a_0}].$$

Now, $$2u\,[\partial_{a_0}]=2\,(a_0+a_1e^{\theta}+a_2e^{2\theta}+\ldots.)[\partial_{a_0}]$$
$$=2(a_0\partial_{a_0}+a_1\partial_{a_1}+a_2\partial_{a_2}+\ldots.).$$

Also $$P=4A_0\partial_{a_1}+5A_1\partial_{a_2}+6A_2\partial_{a_3}+\ldots.,$$
$$Q=A_1'\partial_{a_0}+2A_2'\partial_{a_1}+3A_3'\partial_{a_2}+\ldots.,$$

where $$\frac{1}{2}(a_0+a_1e^{\theta}+a_2e^{2\theta}+\ldots.)^2=A_0+A_1e^{\theta}+A_2e^{2\theta}+\ldots.$$

and $$\log(a_0+a_1e^{\theta}+a_2e^{2\theta}+\ldots.)=\log a_0+A_1'e^{\theta}+A_2'e^{2\theta}+\ldots.$$

Equating coefficients, we have

$$A_0=\frac{1}{2}a_0^2,\ A_1=a_0a_1,\ A_2=a_0a_2+\frac{a_1^2}{2},\ldots.$$
$$A_1'=\frac{a_1}{a_0},\ A_2'=\frac{2a_0a_2-a_1^2}{2a_0^2},\ldots.$$

It is easily seen by expanding the logarithm that the general value of A_n' is $(-)^{n+1}\frac{S_n}{n}$ where S_n denotes the sum of the n^{th} powers of the roots of

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots.+a_n.$$

Thus we have shown that if

$$P=2a_0^2\partial_{a_1}+5a_0a_1\partial_{a_2}+(6a_0a_2+3a_1^2)\,\partial_{a_3}$$

and $$Q=\frac{a_1}{a_0}\partial_{a_0}+\frac{2a_0a_2-a_1^2}{a_0^2}\partial_{a_1}+\frac{3a_0^2a_3-3a_0a_1a_2+a_1^3}{a_0^3}\partial_{a_2}+\ldots.,$$

then $$PQ-QP=2\,(a_0\partial_{a_0}+a_1\partial_{a_1}+a_2\partial_{a_2}+\ldots.)=2i.$$

The general formula obtained for $PQ-QP$ is an extension of a result of Capt. MacMahon's, who considers the case in which

$$\phi(u)=\frac{u^m}{m},\ \psi(u)=\frac{u^{m'}}{m'}.$$

When $\phi(u)$ and $\psi(u)$ have these values, the general formula becomes

$$PQ-QP=e^{(n+n')\theta}\left\{\left(\lambda'+\mu'n+\mu'\frac{d}{d\theta}\right)\left(\frac{\lambda u^{m+m'-1}}{m}+\mu u^{m+m'-2}\frac{du}{d\theta}\right)\right.$$
$$\left.-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right\}[\partial_{a_0}].$$

But
$$\left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left(\frac{\lambda}{m} u^{m+m'-1} + \mu u^{m+m'-2} \frac{du}{d\theta}\right)$$
$$= \left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left(\frac{\lambda}{m} + \frac{\mu}{m+m'-1} \frac{d}{d\theta}\right) u^{m+m'-1}.$$
Consequently
$$PQ - QP = e^{(n+n')\theta} \left\{ \left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left(\frac{\lambda}{m} + \frac{\mu}{m+m'-1} \frac{d}{d\theta}\right) \right.$$
$$\left. - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right\} u^{m+m'-1} [\partial_{a_0}].$$
In Capt. MacMahon's notation
$$P = (m, \lambda, \mu, n), \quad Q = (m', \lambda', \mu', n');$$
in our notation
$$P = e^{n\theta}\left(\lambda + \mu \frac{d}{d\theta}\right) \frac{u^m}{m} [\partial_{a_0}],$$
$$Q = e^{n'\theta}\left(\lambda' + \mu' \frac{d}{d\theta}\right) \frac{u^{m'}}{m'} [\partial_{a_0}].$$
If now we write
$$PQ - QP = e^{(n+n')\theta}\left(\lambda_1 + \mu_1 \frac{d}{d\theta}\right) \frac{u^{m+m'-1}}{m+m'-1} [\partial_{a_0}],$$
which is equivalent to
$$PQ - QP = (m + m' - 1, \lambda_1, \mu_1, n + n'),$$
we have
$$\left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left\{\frac{\lambda}{m}(m + m' - 1) + \mu \frac{d}{d\theta}\right\}$$
$$- \left(\lambda + \mu n' + \mu \frac{d}{d\theta}\right)\left\{\frac{\lambda'}{m'}(m + m' - 1) + \mu' \frac{d}{d\theta}\right\} = \lambda_1 + \mu_1 \frac{d}{d\theta}.$$
Hence we obtain
$$\lambda_1 = (m + m' - 1)\left\{\frac{\lambda}{m}(\lambda' + \mu' n) - \frac{\lambda'}{m'}(\lambda + \mu n')\right\},$$
$$\mu_1 = \mu\mu'(n - n') + \frac{\lambda\mu'}{m}(m' - 1) - \frac{\lambda'\mu}{m'}(m - 1).$$
This agrees with Capt. MacMahon's result, a statement of which was given in Lecture XX.

Let Q be a reversor to the operator $P = \lambda a^m \partial_b + (\ldots.)\,\partial_c + (\ldots.)\,\partial_d + \ldots.$, and suppose that
$$(PQ - QP)\,F = \varkappa a^{m-1} F,$$
where F is any homogeneous and isobaric function and $\varkappa$ some number depending on its degree and weight. Then $\lambda a Q - \varkappa b$ will be the generator corresponding to Q. In other words, we have to prove that
$$P(\lambda a Q - \varkappa b)\,F = 0 \text{ whenever } PF = 0.$$

Now, by hypothesis, $Pa = 0$, $Pb = \lambda a^m$, and when $PF = 0$,

$$PQF = \varkappa a^{m-1} F.$$

Thus,

$$\begin{aligned} P(\lambda a Q - \varkappa b) F &= \lambda a PQF - \varkappa F . Pb \\ &= \lambda \varkappa a^m F - \lambda \varkappa a^m F = 0 . \end{aligned}$$

As an example, consider the case of the reversor $\frac{d}{dx}$ in the theory of reciprocants. Here

$$P = V, \ \lambda = 2, \ m = 2;$$

and since

$$\left(V \frac{d}{dx} - \frac{d}{dx} V\right) F = 2\mu a F,$$

we have $\varkappa = 2\mu$. Hence the corresponding generator is $2\left(a \frac{d}{dx} - \mu b\right)$; or, disregarding the numerical factor 2, we may take $a \frac{d}{dx} - \mu b$ for the generator in question, which is usually denoted by the letter G.

We may also write G in the equivalent form

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \ldots ,$$

which it is sometimes more convenient to use.

I shall now show that

$$\Omega G - G\Omega = aw - b\Omega,$$

where w is the weight of the operand.

It is very easily seen that

$$\begin{aligned} \Omega(ac - b^2) &= 0, \\ \Omega(ad - bc) &= 2(ac - b^2), \\ \Omega(ae - bd) &= 3(ad - bc), \\ \Omega(af - be) &= 4(ae - bd), \\ &\ldots\ldots\ldots \end{aligned}$$

Hence it follows, by a direct and very simple calculation, that

$$\Omega G - G\Omega = 2(ac - b^2)\partial_c + 3(ad - bc)\partial_d + 4(ae - bd)\partial_e + \ldots .$$

But, since

$$b\partial_b + 2c\partial_c + 3d\partial_d + 4e\partial_e + \ldots = w,$$

and

$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots = \Omega,$$

$$aw - b\Omega = 2(ac - b^2)\partial_c + 3(ad - bc)\partial_d + 4(ae - bd)\partial_e + \ldots .$$

Consequently

$$\Omega G - G\Omega = aw - b\Omega.$$

The use of this formula will be seen in a subsequent lecture.

We may also prove an analogous theorem relating to the invariant generator $a \frac{d}{dx} - \nu b$, which we shall call G'.

Let the operand be F, a homogeneous and isobaric function of degree i and weight w. Then VF is of degree $i+1$ and weight $w-1$; its ν is therefore

$$3(i+1)+2(w-1)=\nu+1.$$

Thus, $$(VG'-G'V)F=\left\{V\left(a\frac{d}{dx}-\nu b\right)-\left(a\frac{d}{dx}-\nu b-b\right)V\right\}F$$
$$=a\left(V\frac{d}{dx}-\frac{d}{dx}V\right)F-\nu(Vb-bV)F+bVF.$$

But $$\left(V\frac{d}{dx}-\frac{d}{dx}V\right)F=2\mu aF=2(3i+w)aF$$

and $$VbF=bVF+2a^2F.$$

Consequently
$$VG'-G'V=2(3i+w)a^2F-2\nu a^2F+bVF$$
$$=2(3i+w-\nu)a^2F+bVF$$
$$=-2wa^2F+bVF.$$

It is perhaps worthy of notice that if I is an invariant of weight w and R a pure reciprocant, also of weight w, then

$$\Omega GI=awI \text{ and } VG'R=-2a^2wR;$$

whereas $$\Omega G'I=0 \quad\text{and}\quad VGR=0.$$

LECTURE XXVII.

I should like to make a momentary pause in the development of the theory which now engages our attention and to revert to the proof of Cayley's theorem for the enumeration of linearly independent invariants contained in Lecture XI and expressed by the formula $(w;\ i,\ j)-(w-1;\ i,\ j)$.

Since that proof was written out I have endeavored to obtain one that might be capable of being extended to the supposed analogous theorem, regarding pure reciprocants, expressed by the formula $(w;\ i,j)-(w-1;\ i+1,\ j)$, but all my efforts and those of another and most skilful algebraist in this direction have hitherto proved ineffectual.

In aiming at this object, however, I obtained a second proof of Cayley's theorem less compendious than the previous one, and subject to the drawback that it assumes the law of Reciprocity, but which possesses the advantage over it of being more direct and looking the question, so to say, more squarely in the face. The forms of thought employed in it seem to me too peculiar and precious

The operators which we shall employ, viz. Ω and Ω', are defined by the equations

$$\Omega = a_0\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots ,$$
$$\Omega' = \qquad a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots .$$

The first of these is of course an equivalent, but for present purposes more convenient, form of $a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$, the ordinary invariant annihilator Ω $\left(\text{as will be evident on writing } a_0 = a,\ a_1 = \frac{b}{1},\ a_2 = \frac{c}{1.2}, \ldots\right)$; the second of them, Ω', is merely Ω deprived of its first term.

We may now give the following enunciation of the theorem to be proved: *If ϕ is the most general gradient of its type, $\Omega\phi$ is also the most general gradient of its type whenever $E\phi$ is not negative.* In other words, we shall prove that, subject to the condition stated above, $\Delta\Omega\phi = 0$ whenever $\Delta\phi = 0$. This is equivalent to Cayley's Theorem on the number of linearly independent invariants. For the number of forms of the same type as ϕ, and subject to annihilation by Ω, is

$$N\phi - N\Omega\phi + \Delta\Omega\phi ;$$

and Cayley's Theorem states that the number of such forms is $N\phi - N\Omega\phi$, which will be the case when

$$\Delta\Omega\phi = 0 .$$

The theorem of Reciprocity enables us to dispense with the discussion of those cases in which the extent j is greater than the degree i. For since (see *American Journal of Mathematics*, Vol. I, p. 91) the number of linearly independent invariants for the type $w;\ j,\ i$ is the same as for the type $w;\ i,\ j$, we can substitute the first of these types for the second, using ψ, whose type is $w;\ j,\ i$, instead of ϕ, whose type is $w;\ i,\ j$. Thus we have

$$N\psi - N\Omega\psi + \Delta\Omega\psi = N\phi - N\Omega\phi + \Delta\Omega\phi .$$

But by Ferrers' proof of Euler's Theorem (*vide* A Constructive Theory of Partitions, Vol. V, No. 3 of this Journal),

$$N\psi = N\phi \text{ and } N\Omega\psi = N\Omega\phi .$$

It obviously follows that

$$\Delta\Omega\psi = \Delta\Omega\phi .$$

Cases for which the extent is greater than the degree may therefore be made to depend on those for which the degree is greater than the extent. Hence Cayley's Theorem depends on the proof that $\Delta\Omega\phi = 0$ when $i \mathrel{=\!\!>} j$ and $ij \mathrel{=\!\!>} 2w$.

In the course of the demonstration, the following Lemma will be used:

If $\quad T\phi = w;\ i,\ j$ and $T\psi = ij - w;\ i,\ j$, then $N\phi = N\psi$.

The types of the two gradients we are now considering may be said to be *complementary*, and then the Lemma may be enunciated in words as follows:

The denumerants of two gradients are equal when the types of the gradients are complementary.

The proof consists in showing that to each term of the type $w;\ i,\ j$ there corresponds a term of the type $ij - w;\ i,\ j$. Let $a_0^{\lambda_0} a_1^{\lambda_1} a_2^{\lambda_2} \dots a_j^{\lambda_j}$ be any term of the type $w;\ i,\ j$; then

$$w = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \dots + j\lambda_j$$

and

$$i = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_j.$$

Writing the suffixes of the letters $a_0,\ a_1,\ a_2, \dots a_j$ in reverse order, everything else being kept unchanged, we obtain the term $a_j^{\lambda_0} a_{j-1}^{\lambda_1} a_{j-2}^{\lambda_2} \dots a_0^{\lambda_j}$, whose weight we will call w'. Then

$$\begin{aligned} w' &= j\lambda_0 + (j-1)\lambda_1 + (j-2)\lambda_2 + \dots + \lambda_{j-1} \\ &= j(\lambda_0 + \lambda_1 + \lambda_2 + \dots + \lambda_j) - (\lambda_1 + 2\lambda_2 + 3\lambda_3 + \dots + j\lambda_j) \\ &= ij - w. \end{aligned}$$

The degree of the transformed term is still i, and its extent is still j, while its weight has become $ij - w$; its type is therefore complementary to that of the original term. Hence to each term of any given type there corresponds a term of the complementary type, and consequently the total number of possible terms (*i. e.* the Denumerant) for each type is the same.

By means of this Lemma it can be shown that $\Delta\Omega\phi = 0$ when $E\phi = -1$.

Let $$T\phi = w;\, i,\, j \text{ where } ij - 2w = -1;$$

then, since $T\Omega\phi = w - 1;\ i,\ j$, the types $T\phi$ and $T\Omega\phi$ are complementary (the sum of the weights being $w + w - 1 = ij$).

It follows from the Lemma that the Denumerants of ϕ and $\Omega\phi$ are equal.

Hence $$\Delta\Omega\phi = 0.$$

For if not, the number of independent terms in $\Omega\phi$ being less than the denumerant of $\Omega\phi$, will also be less than its equal, the denumerant of ϕ, and therefore there will be one or more invariants of the type $w;\, i,\, j$ for which the excess is negative. Since this is known to be impossible, we must have

$$\Delta\Omega\phi = 0.$$

We next prove that, in all cases for which $i => w$, the number of linearly independent invariants of the type $w;\, i,\, j$ is correctly given by the formula

$$(w;\, i,\, j) - (w - 1;\, i,\, j),$$

which is equivalent (as we showed at the beginning of Lecture XV) to

$$(w; w, j) - (w - 1; w, j),$$

or, what is the same thing, to the coefficient of $a^w x^w$ in the expansion of

$$F = \frac{1 - x}{(1 - a)(1 - ax)(1 - ax^2)(1 - ax^3) \ldots . (1 - ax^j)}.$$

Let the expansion of

$$G = \frac{1 - x}{(1 - ax)(1 - ax^2)(1 - ax^3) \ldots . (1 - ax^j)}$$

be $$1 + (a - 1)x + A_2 x^2 + \ldots . + A_w x^w + \ldots .$$

The expansion of F is obtained by multiplying that of G by the infinite geometrical series $1 + a + a^2 + a^3 + \ldots .$

But we only require the coefficient of $a^w x^w$ in the expansion of F, so that we need only retain the portion

$$A_w x^w (1 + a + a^2 + \ldots . + a^w)$$

of the above product instead of its complete expression.

It is of importance to notice here that A_w, which is independent of x, cannot contain any higher power of a than a^w. (That this is so will be evident from the constitution of the fraction G, for clearly no power of a in the expansion of G can be associated with a lower power of x.) Thus we see that

$$A_w = \alpha a^w + \beta a^{w-1} + \gamma a^{w-2} + \ldots . + \varkappa a + \lambda,$$

and consequently

$$\begin{aligned} &A_w x^w (1 + a + a^2 + \ldots . + a^w) \\ &= \ldots . + a^w x^w (\alpha + \beta + \gamma \ldots . + \varkappa + \lambda) + \ldots . \end{aligned}$$

Hence the coefficient of $a^w x^w$ in the expansion of F is

$$\alpha + \beta + \gamma + \ldots . + \varkappa + \lambda,$$

which is the value assumed by A_w when in it we write $a = 1$. Call this value A'_w, and let the value of G when $a = 1$ be denoted by G'. Then A'_w is the coefficient of x^w in

$$G' = \frac{1}{(1 - x^2)(1 - x^3) \ldots . (1 - x^j)}.$$

Hence we see that, when $i => w$, the value of $(w; i, j) - (w - 1; i, j)$ is the total number of ways in which w can be made up of the parts $2, 3, \ldots . j$.

We have yet to show that this number is the same as that of the linearly independent invariants of the type $w; i, j$ when $i => w$.

This follows from the known theorem that every invariant is either a rational integral function of the Protomorphs $a, P_2, P_3, \ldots . P_j$ (meaning the

but we may at once dismiss those cases in which $i = > w$, and (assuming the theorem to have been proved for Quantics of order inferior to j) those in which $i < j$, for these depend on the truth of the theorem for a Quantic of order i.

It remains, then, to prove that, when $ij - 2w = > 0$, $\Delta\Omega\,[w;\, i, j] = 0$ for values of i inferior to w, but not inferior to j. This may be effected as follows: Let ϕ be the most general gradient of the type $w;\, i+1, j$, and suppose

$$\phi = P + Qa + Ra^2 + Sa^3,$$

where P, Q and R do not contain the letter a, though S may do so. Then, writing

$$\phi_1 = Q + Ra + Sa^2,$$

ϕ_1 is the most general gradient of the type $w;\, i, j$.

Now, if $\Omega = a\partial_b + b\partial_c + c\partial_d + \ldots$, and $\Omega' = b\partial_c + c\partial_d + \ldots$, we have

$$(1) \qquad \Omega\phi = \Omega' P + \left(\Omega' Q + \frac{dP}{db}\right)a + \left(\Omega' R + \frac{dQ}{db}\right)a^2 + \left(\Omega S + \frac{dR}{db}\right)a^3$$

and
$$\Omega\phi_1 = \Omega' Q + \left(\Omega' R + \frac{dQ}{db}\right)a + \left(\Omega S + \frac{dR}{db}\right)a^2.$$

Confining our attention for the present to $\Omega\phi_1$, it is clear that if no linear relations exist among the coefficients of $\Omega' R$ (*i. e.* if $\Delta\Omega' R = 0$) the coefficients of $\Omega' Q$ are not connected with those of $\Omega' R + \frac{dQ}{db}$ by any linear relation. For the coefficient of each term of $\Omega' R + \frac{dQ}{db}$ is the sum of a single coefficient of Q and an independent linear function of the coefficients of R. Moreover, obviously the coefficients of $\Omega' Q$ are unconnected with those of $\Omega S + \frac{dR}{db}$.

If, then, the coefficients of $\Omega' Q$ are not related *inter se* (*i. e.* if $\Delta\Omega' Q = 0$), we have

$$(2) \qquad \Delta\Omega\phi_1 = \Delta\left\{\left(\Omega' R + \frac{dQ}{db}\right)a + \left(\Omega S + \frac{dR}{db}\right)a^2\right\}.$$

Looking now to the expression (1) for $\Omega\phi$, we see immediately from (2) that any linear relation subsisting between the coefficients of $\Omega\phi_1$ will also subsist between those of $\Omega\phi$, and therefore that $\Delta\Omega\phi_1$ is not greater than $\Delta\Omega\phi$.

If, then, $\Delta\Omega\phi = 0$, it follows that $\Delta\Omega\phi_1 = 0$, provided that both the supplementary conditions $\Delta\Omega' Q = 0$ and $\Delta\Omega' R = 0$ are also satisfied.

Now, since $\phi_1 = Q + Ra + Sa^2$ is the most general gradient of the type $w;\, i, j$,

Q will be the most general gradient of the type $w - i;\, i, j - 1$

and

R " " " " " " " " " $w - i + 1;\, i - 1, j - 1$,

when in Q and R we change $b, c, d, \ldots$ into $a, b, c, \ldots$ This change converts $\Omega' = b\partial_c + c\partial_d + \ldots$ into $\Omega = a\partial_b + b\partial_c + \ldots$ Hence the conditions $\Delta\Omega' Q = 0$ and $\Delta\Omega' R = 0$ are respectively equivalent to

$$\Delta\Omega\,[w - i;\, i, j - 1] = 0 \text{ and } \Delta\Omega\,[w - i + 1;\, i - 1, j - 1] = 0.$$

Supposing these supplementary conditions to be satisfied, what we have proved is that when

$$\Delta\Omega\,[w;\, i + 1, j] = 0 \ (i.\,e.\ \Delta\Omega\phi = 0),$$

then also $\qquad \Delta\Omega\,[w;\, i, j] = 0 \ (i.\,e.\ \Delta\Omega\phi_1 = 0).$

Now,

$$\begin{aligned}
T\phi &= w;\, i + 1, j, &&\text{so that } E\phi = (i + 1)j - 2w = (ij - 2w) + j,\\
TQ &= w - i;\, i, j - 1, &&\text{" " } EQ = i(j - 1) - 2(w - i) = (ij - 2w) + i,\\
TR &= w - i + 1;\, i - 1, j - 1, &&\text{so that } ER = (i - 1)(j - 1) - 2(w - i + 1)\\
& && \qquad = (ij - 2w) + i - j - 1.
\end{aligned}$$

Thus, when $ij - 2w => 0$ and $i => j$,

$E\phi$ and EQ are both positive.

ER is in general $=> 0$, but in the special case where $ij - 2w = 0$ and $i = j$, we have $ER = -1$. Except in this case (which gives us no trouble, since we have seen that $\Delta\Omega R = 0$ in consequence of $ER = -1$), we have never to deal with a type of which the excess is negative.

Hence, if we assume Cayley's Theorem to have been proved for all extents up to $j - 1$ inclusive, we have

$$\Delta\Omega\,[w - i;\, i, j - 1] = 0$$

and $\qquad \Delta\Omega\,[w - i + 1;\, i - 1, j - 1] = 0$

($i.\,e.$ the two supplementary conditions are satisfied).

We wish to extend the theorem to the extent j.

Subject to the conditions $i => j$ and $ij - 2w => 0$, we have

$$\Delta\Omega\,[w;\, i, j] = 0 \text{ if } \Delta\Omega\,[w;\, i + 1, j] = 0.$$

But we need consider no value of i greater than w, as we have proved that

$$\Delta\Omega\,[w;\, w, j] = 0 = \Delta\Omega\,[w;\, w + x, j];$$

therefore

$$\begin{aligned}
&\Delta\Omega\,[w;\, w - 1, j] = 0,\\
&\Delta\Omega\,[w;\, w - 2, j] = 0,\\
&\cdots\cdots\cdots\cdots\\
&\Delta\Omega\,[w;\, j, j] \quad = 0.
\end{aligned}$$

As previously shown, the theorem is true for all values of i inferior to j if it is true for all Quantics of inferior order. Thus the theorem is true for a Quantic of order j and for every value of i if it is true for all Quantics of order

inferior to j. But it is true for the Quadric (where $j = 2$);* therefore also for the Cubic ($j = 3$); therefore also for the Quartic ($j = 4$), and so universally. Hence the theorem to be proved is demonstrated.

LECTURE XXVIII.

We now resume the theory of Principiants and proceed to prove the important theorem that every Principiant is either simply an invariant in respect to a known series of pure reciprocants, which we call $A, B, C, D, \ldots$, or else becomes such an invariant when multiplied by a^{w-i}, where w is the weight and i the degree of the Principiant in question. Using the letter M to denote the pure reciprocant $ac - \frac{5}{4} b^2$, and G the ordinary eductive generator,

$$4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + 7(af - be)\partial_e + \ldots$$

(which, it will be remembered, is only another form of $a\frac{d}{dx} - \mu b$, with the advantage of the μ being suppressed, *i. e.* only implicitly contained), we obtain in succession the values of $A, B, C, D, \ldots$ from the following equations:

$$\begin{aligned}
5A &= GM,\\
6B &= GA,\\
7C &= GB - MA,\\
8D &= GC - 2MB,\\
9E &= GD - 3MC,\\
&\ldots\ldots\ldots
\end{aligned}$$

On performing the calculations indicated by these equations we shall find

$$\begin{aligned}
A &= a^2d - 3abc + 2b^3,\\
B &= a^3e - 2a^2c^2 - \frac{7}{2}a^2bd + \frac{17}{2}ab^2c - 4b^4,\\
C &= a^4f - 5a^3cd - 4a^3be + 13a^2bc^2 + \frac{45}{4}a^2b^2d - \frac{103}{4}ab^3c + \frac{19}{2}b^5,\\
D &= a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3 + \text{terms involving } b,\\
E &= a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d + \text{terms involving } b.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

* When $j = 2$ the condition $ij => 2w$ becomes identical with $i => w$; but we have already seen that the theorem is true whenever $i => w$.

Thus, *ex. gr.*

$$8D_0 = G_0(a^4f - 5a^3cd) - 2ac(a^3e - 2a^2c^2)$$
$$= -25a^4d^3 - 30a^3c(ae - 2c^2) + 8a^3(a^2g - 2ace - c^3) - 2ac(a^3e - 2a^2c^3);$$

whence $$D_0 = a^5g - \frac{25}{8}a^4d^3 - 6a^4ce + 7a^3c^3.$$

Again, $$9E_0 = G_0\left(a^5g - \frac{25}{8}a^4d^3 - 6a^4ce + 7a^3c^3\right) - 3ac(a^4f - 5a^3cd)$$
$$= 5ad(-6a^4e + 21a^3c^3) - \frac{75}{2}(ae - 2c^2)a^4d - 42(af - 2cd)a^4c$$
$$+ 9(a^2h - 2acf - 2c^2d)a^4 - 3ac(a^4f - 5a^3cd)$$

gives $$E_0 = a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d.$$

Similarly, from the known values of D_0 and E_0 we may deduce that of the next letter, F_0, and so on to any extent.

It may be noticed that each of the pure reciprocants A, B, C, D, ... can be determined without ambiguity, by means of the annihilator V, when the portions of them, A_0, B_0, C_0, D_0, ... independent of b are known.

For suppose R and R' to be two reciprocants, of weight w, for each of which the terms independent of b are the same. Then their difference is divisible by b. Let

$$R - R' = b\phi; \text{ then } V(b\phi) = 0; \text{ i. e. } 2a^2\phi + bV\phi = 0.$$

Hence ϕ is divisible by b, and $R - R'$ is divisible by b^2; say $R - R' = b^2\psi$. Then
$$V(b^2\psi) = 4a^2b\psi + b^2V\psi = 0,$$
showing that ψ is divisible by b, and $R - R'$ by b^3.

By continually reasoning in this manner, we prove that $R - R'$ must be divisible by b^w; and then the remaining factor (being of weight 0) is necessarily of the form λa^θ, where λ and θ are numerical constants. Thus

$$R - R' = \lambda a^\theta b^w, \text{ and consequently } V(\lambda a^\theta b^w) = 0.$$

This is impossible unless $\lambda = 0$, when the two reciprocants R, R' become equal, showing that there cannot be two different reciprocants for which the terms independent of b are the same. When, therefore, the terms which do not involve b of any pure reciprocant are known, the complete expression of that reciprocant can be determined without ambiguity.

Each reciprocant of the series A, B, C, D, ... possesses the property of being, so to say, an Invariant relative to the one which precedes it, meaning that the operation of $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots.$ on any letter gives (to a

Now, $\quad 7C = GB - MA;$

so that $\quad 7\Omega C = \Omega GB - A\Omega M - M\Omega A.$

But, since $\quad \Omega M = \Omega\left(ac - \frac{5b^2}{4}\right) = -\frac{ab}{2}$ and $\Omega A = 0$,

$$7\Omega C = \Omega GB + \frac{ab}{2} A = 7aB.$$

Thus $\quad \Omega C = 2B \times \frac{a}{2}.$

We may, in exactly the same way, prove that

$$\Omega D = 3C \times \frac{a}{2},$$

$$\Omega E = 4D \times \frac{a}{2},$$

and so on to any extent.

In the following inductive proof it will be convenient to denote the letters

$$A,\ B,\ C,\ D,\ E, \ldots .$$

by

$$u_0,\ u_1,\ u_2,\ u_3,\ u_4, \ldots ,$$

and then the theorem to be proved is that

$$\Omega u_n = n u_{n-1} \times \frac{a}{2}.$$

When this notation is used, the law of successive derivation which defines the capital letters is expressed by the equation

(1) $\quad (n+7)\, u_{n+2} - G u_{n+1} + (n+1) M u_n = 0,$

where G is the generator $4(ac - b^2)\,\partial_b + 5(ad - bc)\,\partial_c + \ldots .$, and $M = ac - \frac{5b^2}{4}$.

Operating with Ω on the above equation, we obtain

(2) $\quad (n+7)\, \Omega u_{n+2} - \Omega G u_{n+1} + (n+1)(M\Omega u_n + u_n \Omega M) = 0.$

Now, the weights of $u_0, u_1, u_2, \ldots .$ are $3, 4, 5, \ldots .$ respectively, and consequently the operation of

$$\Omega G - G\Omega = wa - b\Omega$$

on u_{n+1} (whose weight is $n+4$) gives

$$(\Omega G - G\Omega)\, u_{n+1} = (n+4)\, a u_{n+1} - b\Omega u_{n+1}.$$

Or, assuming that $\Omega u_x = x u_{x-1} \times \frac{a}{2}$ for all values of x as far as $n+1$ inclusive (it has previously been shown that $\Omega B = A \times \frac{a}{2}$ and $\Omega C = 2B \times \frac{a}{2}$, so that the theorem is true for $x = 1$ and $x = 2$),

$$\begin{aligned} \Omega G u_{n+1} &= G\Omega u_{n+1} + (n+4)\, a u_{n+1} - b\Omega u_{n+1} \\ &= (n+1)\, G\left(\frac{a}{2} u_n\right) + (n+4)\, a u_{n+1} - (n+1)\frac{ab}{2} u_n . \end{aligned}$$

But $\left(\text{remembering that } G \text{ does not operate on } a, \text{ so that } G.\frac{a}{2}u_n = \frac{a}{2}Gu_n\right)$ we have, in virtue of equation (1),

$$G\left(\frac{a}{2}u_n\right) = \frac{a}{2}\left\{(n+6)u_{n+1} + nMu_{n-1}\right\}.$$

Hence it follows that

$$\begin{aligned}\Omega Gu_{n+1} &= \frac{n+1}{2}a\{(n+6)u_{n+1} + nMu_{n-1}\} + (n+4)au_{n+1} - (n+1)\frac{ab}{2}u_n \\ &= \frac{(n+2)(n+7)}{2}au_{n+1} + \frac{n(n+1)}{2}aMu_{n-1} - (n+1)\frac{ab}{2}u_n.\end{aligned}$$

On substituting this in (2) we obtain

$$\begin{aligned}&(n+7)\left\{\Omega u_{n+2} - (n+2)\frac{a}{2}u_{n+1}\right\} \\ &+ (n+1)M\left\{\Omega u_n - n\frac{a}{2}u_{n-1}\right\} \\ &+ (n+1)u_n\left\{\Omega M + \frac{ab}{2}\right\} = 0.\end{aligned}$$

This reduces to

$$\Omega u_{n+2} = (n+2)\frac{a}{2}u_{n+1}.$$

For, according to the assumption previously made in the course of the demonstration,

$$\Omega u_n = n\frac{a}{2}u_{n-1};$$

so that the second term vanishes; and the third term vanishes because

$$\Omega M = \Omega\left(ac - \frac{5b^2}{4}\right) = -\frac{ab}{2}.$$

We have therefore proved that if the theorem is true for Ωu_κ, when κ has any value up to $n+1$ inclusive, it is also true for Ωu_{n+2}. But the theorem holds for $\kappa = 1$, and for $\kappa = 2$. It therefore holds universally for any positive integer value of κ.

Recalling the known values of the reciprocants $M, A, B, C, D, \ldots.$ we observe that their principal terms are $ac, a^2d, a^3e, a^4f, a^5g, \ldots.$, where it is to be noticed that the most advanced of the small letters in the expression for any capital letter occurs only in the first degree multiplied by a power of a. In other words, $M, A, B, C, D, \ldots.$ form a series of Protomorphs, and consequently every Pure Reciprocant can, as we have already seen (vide *American Journal of Mathematics*, Vol. IX, p. 35), be expressed as a function of $a, M, A, B, C, D, \ldots.$ rational in all of them and integral in all except a.

But it is further to be noticed that whereas

a is of	degree	1	and weight	0,
M " "	"	2	" "	2,
A " "	"	3	" "	3,
B " "	"	4	" "	4,

and in fact that every capital letter is of equal weight and degree.

From this it will follow that every Pure Reciprocant will be the product of a power of a into a function of the capital letters alone.

For let i be the degree and w the weight of any pure reciprocant expressed in terms of a, M, A, B, C, ..., and suppose one of its terms to be

$$a^\eta M^\theta A^\kappa B^\lambda C^\mu \ldots ;$$

then $\qquad \eta + 2\theta + 3\kappa + 4\lambda + 5\mu + \ldots = i$

and $\qquad 2\theta + 3\kappa + 4\lambda + 5\mu + \ldots = w.$

Hence $\qquad \eta = i - w,$

which is the same for every term of the pure reciprocant in question. Thus each term contains a^{i-w} as a factor, and the reciprocant is of the form

$$a^{i-w}\Phi(M, A, B, C, D, \ldots).$$

Let us now consider any Principiant P; since P is a pure reciprocant, we must have $\qquad P = a^{i-w}\Phi(M, A, B, C, D, \ldots).$

But Principiants are subject to annihilation by Ω, and consequently $\Omega P = 0$, which gives

$$\frac{d\Phi}{dM}\Omega M + \frac{d\Phi}{dA}\Omega A + \frac{d\Phi}{dB}\Omega B + \frac{d\Phi}{dC}\Omega C + \ldots = 0.$$

On writing for $\qquad \Omega M, \quad \Omega A, \quad \Omega B, \quad \Omega C, \ldots$

their values $\qquad -b\times\frac{a}{2}, \quad 0, \quad A\times\frac{a}{2}, \quad 2B\times\frac{a}{2}, \ldots$

we obtain

$$\frac{a}{2}(-b\partial_M + A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\Phi = 0.$$

From this it would follow that Φ is an invariant in the two sets of letters

$$-b,\ M \text{ and } A,\ B,\ C,\ D, \ldots ;$$

but it is easy to see that it is an invariant in the latter set exclusively. For M and A, B, C, D, ... being all of them pure reciprocants,

$$\Phi \text{ and } \partial_M\Phi,\ \partial_B\Phi,\ \partial_C\Phi,\ \partial_D\Phi, \ldots,$$

which are functions of M, A, B, C, ... exclusively, must also be pure reciprocants.

If, then, we operate with V on

$$(-b\partial_M + A\partial_B + 2B\partial_C + 3C\partial_D, \ldots .)\Phi = 0,$$

we shall find $V(-b\partial_M)\Phi = 0$ (every other term being annihilated by V). Thus

$$V(b\partial_M)\Phi = (\partial_M\Phi)\,Vb = 2a^2\partial_M\Phi = 0,$$

and consequently $\partial_M\Phi = 0$. Hence

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots .)\Phi = 0.$$

The equation $\partial_M\Phi = 0$ shows that M does not appear in the expression for any principiant in terms of the capital letters, while

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots .)\Phi = 0$$

shows that Φ is an invariant in $A, B, C, D, \ldots .$

We have thus shown that every invariant of

$$(A, B, C, \ldots .)(x, y)^j$$

is a principiant, and conversely that every principiant is an invariant of

$$(A, B, C, \ldots .)(x, y)^j,$$

or such an invariant multiplied by a power of a.

LECTURE XXIX.

From the theorem that every Principiant is (to a power of a près) an Invariant in the reciprocantive elements $A, B, C, \ldots .$ we readily deduce its correlative in which, everything else remaining unchanged, the *reciprocantive* elements $A, B, C, \ldots .$ are replaced by a set of *invariantive* elements which we call $A_0, A_1, A_2, \ldots .$ The equations connecting the new elements with the old ones are as follows:

$$A_0 = A,$$
$$A_1 = B - \left(\frac{b}{2}\right)A,$$
$$A_2 = C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A,$$
$$A_3 = D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2 B - \left(\frac{b}{2}\right)^3 A,$$
$$A_4 = E - 4\left(\frac{b}{2}\right)D + 6\left(\frac{b}{2}\right)^2 C - 4\left(\frac{b}{2}\right)^3 B + \left(\frac{b}{2}\right)^4 A,$$

. . . .

We have, in the first place, to prove that $A_0, A_1, A_2, \ldots$ are all of them invariants in the small letters $a, b, c, \ldots$. This is an immediate consequence of the identities

$$\Omega A = 0,$$
$$\Omega B = A \times \frac{a}{2},$$
$$\Omega C = 2B \times \frac{a}{2},$$
$$\cdots\cdots\cdots\cdots$$

established in the preceding Lecture, coupled with the fact that $\Omega b = a$. Thus

$$\Omega A_0 = \Omega A = 0,$$
$$\Omega A_1 = -\frac{b}{2}\,\Omega A + \left(\Omega B - A \times \frac{a}{2}\right) = 0,$$
$$\Omega A_2 = \left(\frac{b}{2}\right)^2 \Omega A - 2\left(\frac{b}{2}\right)\left(\Omega B - A \times \frac{a}{2}\right) + \left(\Omega C - 2B \times \frac{a}{2}\right) = 0;$$

and in general, writing the equation which gives A_n in the form

$$A_n = \left(-\frac{b}{2}\right)^n A + n\left(-\frac{b}{2}\right)^{n-1} B + \frac{n(n-1)}{1.2}\left(-\frac{b}{2}\right)^{n-2} C + \frac{n(n-1)(n-2)}{1.2.3}\left(-\frac{b}{2}\right)^{n-3} D + \ldots,$$

and operating on it with Ω, we find

$$\Omega A_n = \left(-\frac{b}{2}\right)^n \Omega A + n\left(-\frac{b}{2}\right)^{n-1}\left(\Omega B - A \times \frac{a}{2}\right) + \frac{n(n-1)}{1.2}\left(-\frac{b}{2}\right)^{n-2}\left(\Omega C - 2B \times \frac{a}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}\left(-\frac{b}{2}\right)^{n-3}\left(\Omega D - 3C \times \frac{a}{2}\right) + \ldots$$

$= 0$ (each term vanishing separately).

We next observe that

$(A_0, A_1, A_2, \ldots)(x, y)^j$, being equal to $(A, B, C, \ldots)\left(x - \frac{b}{2}y, y\right)^j$, is a linear transformation of $(A, B, C, \ldots)(x, y)^j$,

and that the determinant of the transformation $\begin{vmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{vmatrix}$ is equal to unity.

Hence every invariant in $A_0, A_1, A_2, \ldots$ is equal to the corresponding invariant in $A, B, C, \ldots$, which proves the theorem in question.

Each of the invariantive elements $A_0, A_1, A_2, \ldots$ is, so to say, a *reciprocant* relative to the one which immediately precedes it, just as in the cognate

theorem each of the capital letters $A, B, C, \ldots$ was an *invariant* relative to its antecedent. It is in fact easily seen that

$$VA_0 = 0,$$
$$VA_1 = -A_0a^2,$$
$$VA_2 = -2A_1a^2,$$
$$VA_3 = -3A_2a^2,$$
$$\cdots\cdots\cdots\cdots$$

and in general
$$VA_n = -nA_{n-1}a^2.$$

Thus, for example, if we operate with V on

$$A_3 = D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2B - \left(\frac{b}{2}\right)^3A,$$

remembering that A, B, C, D are pure reciprocants, we shall find

$$VA_3 = -\frac{3}{2}\left\{C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2A\right\}Vb.$$

But
$$C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2A = A_2 \text{ and } Vb = 2a^2;$$

so that
$$VA_3 = -3A_2a^2.$$

In like manner, operating with V on

$$A_n = (A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^n,$$

we obtain
$$VA_n = -\frac{n}{2}(A, B, C, \ldots)\left(-\frac{b}{2}, 1\right)^{n-1}Vb$$
$$= -nA_{n-1}a^2.$$

This property enables us to give a proof (exactly similar to the proof of the cognate theorem in the preceding Lecture) of the theorem that every principiant is expressible as the product of an invariant in $A_0, A_1, A_2, \ldots$ by a suitable power of a. We first observe that, using N to denote $ac - b^2$,

$$N, A_0, A_1, A_2, \ldots$$

form a series of invariantive protomorphs of equal degree and weight.

Hence it follows that any invariant of degree i and weight w can be expressed in the form

$$a^{i-w}\Phi(N, A_0, A_1, A_2, \ldots),$$

and consequently that every Principiant can be expressed in this form, provided only that
$$V\Phi = 0.$$

Substituting for VA_0, VA_1, VA_2, ... their values given above, and at the same time observing that

$$VN = V(ac - b^2) = 5a^2b - 4a^2b = a^2b,$$

we find $\quad V\Phi = a^2(b\partial_N - A_0\partial_{A_1} - 2A_1\partial_{A_2} - 3A_2\partial_{A_3} - \ldots.)\Phi = 0.$

Finally, we prove that Φ does not contain N, but is an invariant in A_0, A_1, A_2, ... alone, by operating with Ω on

$$(b\partial_N - A_0\partial_{A_1} - 2A_1\partial_{A_2} - 3A_2\partial_{A_3} - \ldots.)\Phi = 0,$$

when it is easily seen that every term vanishes except the first, which gives

$$\Omega(b\partial_N\Phi) = \Omega b \times \partial_N\Phi = 0,$$

where, $\Omega b = a$ being different from zero, we must have $\partial_N\Phi = 0$.

The invariants N, A_0, A_1, A_2, ... obey a law of successive derivation similar to that which holds for the reciprocants M, A, B, C, ...

Starting with $N = ac - b^2$ and operating continually with

$$G' = a\frac{d}{dx} - (3i + 2w)b = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c + \ldots.,$$

we shall find

$$\begin{aligned} G'N &= 5A_0, \\ G'A_0 &= 6A_1, \\ G'A_1 &= 7A_2 - NA_0, \\ G'A_2 &= 8A_3 - 2NA_1, \\ G'A_3 &= 9A_4 - 3NA_2, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

and generally $\quad G'A_n = (n+6)A_{n+1} - nNA_{n-1}$.

These equations are exactly analogous to

$$\begin{aligned} GM &= 5A, \\ GA &= 6B, \\ GB &= 7C + MA, \\ GC &= 8D + 2MB, \\ GD &= 9E + 3MC, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

in which $M = ac - \frac{5}{4}b^2$, and GM, GA, GB, ... are the educts of M, A, B, ... obtained by operating with

$$G = a\frac{d}{dx} - (3i + w)b = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + \ldots.$$

It should be noticed that the two generators G and G' are connected by the relation $\quad G' = G - wb,$

where w is the weight of the operand.

Also, that

$$Gb = 4(ac - b^2) = 4N, \text{ and } G'b = 4ac - 5b^2 = 4M.$$

We may easily verify that

$$G'N = 5A_0 = 5(a^2d - 3abc + 2b^3)$$

by operating with $G' = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c$ on $N = ac - b^2$.

To prove that $\quad G'A_0 = 6A_1,$

we operate on $\quad A_0 = A,$

for which the weight is 3, with

$$G' = G - 3b.$$

Thus $\quad G'A_0 = (G - 3b)A = 6B - 3bA = 6A_1.$

For by definition $\quad A_1 = B - \left(\frac{b}{2}\right)A.$

In general, to find $G'A_n$, we have by definition

$$A_n = (A, B, C, \ldots .)\left(-\frac{b}{2}, 1\right)^n,$$

and, since the weight of A_n is $n + 3$,

$$G'A_n = GA_n - (n+3)bA_n.$$

Now,

$$GA_n = G(A, B, C, \ldots .)\left(-\frac{b}{2}, 1\right)^n$$

$$= (GA, GB, GC, \ldots .)\left(-\frac{b}{2}, 1\right)^n - \frac{n}{2}(A, B, C, \ldots .)\left(-\frac{b}{2}, 1\right)^{n-1} Gb.$$

Substituting for GA, GB, GC, ... their known values, and remembering that $Gb = 4N$ and that $(A, B, C, \ldots .)\left(-\frac{b}{2}, 1\right)^{n-1} = A_{n-1}$, we have

$$GA_n = (6B, 7C, 8D, \ldots .)\left(-\frac{b}{2}, 1\right)^n$$

$$+ M(0, A, 2B, 3C, \ldots .)\left(-\frac{b}{2}, 1\right)^n - 2nNA_{n-1}$$

$$= 6(B, C, D, \ldots .)\left(-\frac{b}{2}, 1\right)^n + (0, C, 2D, 3E, \ldots .)\left(-\frac{b}{2}, 1\right)^n$$

$$+ M(0, A, 2B, 3C, \ldots .)\left(-\frac{b}{2}, 1\right)^n - 2nNA_{n-1}.$$

But $\quad (0, C, 2D, 3E, \ldots .)\left(-\frac{b}{2}, 1\right)^n$

$$= nC\left(-\frac{b}{2}\right)^{n-1} + n(n-1)D\left(-\frac{b}{2}\right)^{n-2}$$

$$+ \frac{n(n-1)(n-2)}{1.2}E\left(-\frac{b}{2}\right)^{n-3} + \ldots .$$

$$= n(C, D, E, \ldots .)\left(-\frac{b}{2}, 1\right)^{n-1};$$

and similarly

$$(0,\, A,\, 2B,\, 3C,\, \ldots .)\left(-\frac{b}{2},\, 1\right)^n = n(A,\, B,\, C,\, \ldots .)\left(-\frac{b}{2},\, 1\right)^{n-1} = nA_{n-1}.$$

Hence

$$GA_n = 6(B,\, C,\, D,\, \ldots .)\left(-\frac{b}{2},\, 1\right)^n + n(C,\, D,\, E,\, \ldots .)\left(-\frac{b}{2},\, 1\right)^{n-1} + n(M-2N)A_{n-1}.$$

Now let $U = (A,\, B,\, C,\, \ldots .)(u,\, v)^n;$

then $\dfrac{dU}{du} = n(A,\, B,\, C,\, \ldots .)(u,\, v)^{n-1},$

and $\dfrac{dU}{dv} = n(B,\, C,\, D,\, \ldots .)(u,\, v)^{n-1};$

whence it follows that

(1) $$U = (A,\, B,\, C,\, \ldots .)(u,\, v)^n = u(A,\, B,\, C,\, \ldots .)(u,\, v)^{n-1} + v(B,\, C,\, D,\, \ldots .)(u,\, v)^{n-1}.$$

Similarly, we see that

(2) $$(B,\, C,\, D,\, \ldots .)(u,\, v)^n = u(B,\, C,\, D,\, \ldots .)(u,\, v)^{n-1} + v(C,\, D,\, E,\, \ldots .)(u,\, v)^{n-1}.$$

Writing $u = -\dfrac{b}{2}$ and $v = 1$ in the above equations, and remembering that

$$(A,\, B,\, C,\, \ldots)\left(-\frac{b}{2},\, 1\right)^n = A_n,$$

we obtain immediately from (1)

$$(B,\, C,\, D,\, \ldots .)\left(-\frac{b}{2},\, 1\right)^{n-1} = A_n + \frac{b}{2}A_{n-1},$$

and then (2) gives

$$(C,\, D,\, E,\, \ldots .)\left(-\frac{b}{2},\, 1\right)^{n-1} = \left(A_{n+1} + \frac{b}{2}A_n\right) + \frac{b}{2}\left(A_n + \frac{b}{2}A_{n-1}\right)$$
$$= A_{n+1} + bA_n + \frac{b^2}{4}A_{n-1}.$$

But it has been shown that

$$GA_n = 6(B,\, C,\, D,\, \ldots .)\left(-\frac{b}{2},\, 1\right)^n + n(C,\, D,\, E,\, \ldots .)\left(-\frac{b}{2},\, 1\right)^{n-1} + n(M-2N)A_{n-1}.$$

Hence, by substitution,

$$GA_n = 6\left(A_{n+1} + \frac{b}{2}A_n\right) + n\left(A_{n+1} + bA_n + \frac{b^2}{4}A_{n-1}\right) + n(M-2N)A_{n-1}$$
$$= (n+6)A_{n+1} + (n+3)bA_n + n\left(M + \frac{b^2}{4} - 2N\right)A_{n-1}.$$

Now,

$$G'A_n = GA_n - (n+3)bA_n$$
$$= (n+6)A_{n+1} + n\left(M + \frac{b^2}{4} - 2N\right)A_{n-1},$$

where $$M + \frac{b^2}{4} = ac - \frac{5}{4}b^2 + \frac{b^2}{4} = ac - b^2 = N.$$

Thus $$G'A_n = (n+6)A_{n+1} - nNA_{n-1},$$

which proves the law of successive derivation for the invariantive elements $A_0, A_1, A_2, \ldots$.*

We now proceed to explain the method of transforming a Principiant, given in terms of the small letters $a, b, c, \ldots$, into one expressed in terms of $a, A, B, C, \ldots$.

Remembering that the expressions for

$$A, B, C, D, E, \ldots$$

have for their most advanced small letters

$$d, e, f, g, h, \ldots,$$

and that, in each capital letter, the most advanced letter occurs only in the first degree, multiplied by a power of a, it follows, as an immediate consequence, that we may, by continually substituting for the most advanced letter, eliminate $d, e, f, g, h, \ldots$ from any rational integral function

$$\phi(a, b, c, d, e, f, g, h, \ldots)$$

and thus transform it into another function whose arguments are

$$a, b, c, A, B, C, D, E, \ldots$$

and which is rational in all its arguments, and integral in all of them, with the possible exception of the first argument, a.

But (see Lecture XXVIII) the result of this elimination is known to be

$$a^{i-w}\Phi(A, B, C, D, E, \ldots)$$

in the case where ϕ is a Principiant of known degree i and weight w. Hence b and c must disappear spontaneously during the process of elimination.

This being so, we can give b and c any arbitrary values, without thereby affecting the result, and it will greatly simplify the work to take $b = 0$ and $c = 0$.

It is also permissible to take $a = 1$; for, although the factor a^{i-w} is thereby lost, it can always be restored in the final result because both i and w are known

* The establishment of the scale of relation between the terms of the $A_0, A_1, A_2, \ldots$ series and the above proof of it is due exclusively to Mr. Hammond. J. J. S.

numbers. Now, if we write $a = 1$, $b = 0$, $c = 0$ in the known expressions for $A, B, C, D, \ldots$, we shall find

$$\begin{aligned} A &= d, \\ B &= e, \\ C &= f, \\ D &= g - \tfrac{25}{8} d^2, \\ E &= h - \tfrac{15}{2} de, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Hence we have to eliminate $d, e, f, g, h, \ldots$ between the above equations and

$$P = \phi\,(1, 0, 0, d, e, f, g, h, \ldots),$$

where P stands for the given Principiant. In other words, we have to substitute for

$$\begin{array}{cccccccccc} a, & b, & c, & d, & e, & f, & g & , & h & ,\ldots \\ 1, & 0, & 0, & A, & B, & C, & D + \frac{25}{8}A^2, & & E + \frac{15}{2}AB, & \ldots \end{array}$$

in

$$P = \phi\,(a, b, c, d, e, f, g, h, \ldots).$$

The result of this substitution will be

$$P = \Phi(A, B, C, D, E, \ldots),$$

where, to compensate for the factor lost by taking $a = 1$, we must multiply Φ by a^{i-w}. As an easy example, consider the Principiant which Halphen calls Δ, and for which he obtains the expression

$$\left|\begin{array}{ccccc} b & c & d & e & f \\ a & b & c & d & e \\ -a^2 & 0 & b^2 & 2bc & 2bd + c^2 \\ 0 & a^2 & 2ab & 2ac + b^2 & 2ad + 2bc \\ 0 & 0 & a^2 & 3ab & 3b^2 + 3ac \end{array}\right|.$$

Here the degree $i = 8$ and the weight $w = 8$; so that $i - w = 0$, and no factor has to be restored. On making the substitutions spoken of, the determinant becomes

$$\left|\begin{array}{ccccc} 0 & 0 & A & B & C \\ 1 & 0 & 0 & A & B \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2A \\ 0 & 0 & 1 & 0 & 0 \end{array}\right|,$$

which immediately reduces to $AC - B^2$ by striking out the first three columns and the last three rows.

Of this Principiant we shall have more to say hereafter.

LECTURE XXX.

The method of substituting large letters for small ones will be better understood if we employ it to obtain an expression of the form

$$a^{i-w}\Phi(M, A, B, C, D, E, \ldots .)$$

for any pure reciprocant

$$\phi(a, b, c, d, e, f, g, h, \ldots .)$$

of known degree i and weight w in the small letters.

The transformation is effected by substituting in ϕ for $c, d, e, f, g, h, \ldots .$ their values (which are perfectly definite) in terms of $a, b, M, A, B, C, D, E, \ldots .$ But since b does not appear in the final result, we are at liberty to give it any arbitrary value, and it will be convenient to take $b = 0$, for then (see Lecture XXVIII) we have

$$\begin{aligned}
M &= ac,\\
A &= a^2d,\\
B &= a^3e - 2a^2c^2,\\
C &= a^4f - 5a^3cd,\\
D &= a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3,\\
E &= a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d,\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

There is an additional advantage in taking $b = 0$, viz. that then the values of the *invariants* $N, A_0, A_1, A_2, \ldots .$ (see their definition at the beginning of Lecture XXIX) exactly coincide with those of the *reciprocants* $M, A, B, C, \ldots .$ set forth above. Hence, merely interchanging the capital letters, the same substitutions enable us to express any invariant in terms of $a, N, A_0, A_1, \ldots .$, as well as any reciprocant in terms of $a, M, A, B, \ldots .$

The solution of the above equations will give $\frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots .$ in terms of $\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots .$; but we can, without loss of generality, put $a = 1$, when we shall find

$$\begin{aligned}
a &= 1,\\
b &= 0,\\
c &= M,\\
d &= A,\\
e &= B + 2M^2,\\
f &= C + 5MA,\\
g &= D + \frac{25}{8}A^2 + 6MB + 5M^3,\\
h &= E + \frac{15}{2}AB + 7MC + 6MA^2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The substitution of these values in the pure reciprocant

$$\phi(a, b, c, d, e, f, g, h, \ldots.)$$

will convert it into

$$\Phi(M, A, B, C, D, E, \ldots.).$$

We have written $a = 1$ for the sake of simplicity; but without doing this we have, since ϕ is homogeneous of degree i,

$$\phi(a, 0, c, d, e, \ldots.) = a^i\phi\left(1, 0, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots.\right).$$

Hence, substituting for $\frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots.$ in terms of $\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots.,$

$$\phi(a, 0, c, d, e, \ldots.) = a^i\Phi\left(\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots.\right);$$

or, since $M, A, B, \ldots.$ are of weights $2, 3, 4, \ldots.$, and Φ is of weight w,

$$\phi(a, 0, c, d, e, \ldots.) = a^{i-w}\Phi(M, A, B, \ldots.).$$

Thus, in consequence of writing $a = 1$, the factor a^{i-w} has been lost; but this factor can always be restored, both i and w being known numbers.

When ϕ is a Principiant, M will not appear in the final result, which will be identical with that obtained by the simpler substitutions of the preceding Lecture. If, for example, we substitute for

$$\begin{array}{llllll} a, & b, & c, & d, & e, & f, \\ 1, & 0, & M, & A, & B + 2M^2, & C + 5MA, \end{array}$$

instead of

$$\begin{array}{llllll} 1, & 0, & 0, & A, & B, & C, \end{array}$$

in the determinant expression for Halphen's Δ, previously given, it becomes

$$\begin{vmatrix} 0 & M & A & B + 2M^2 & C + 5MA \\ 1 & 0 & M & A & B + 2M^2 \\ -1 & 0 & 0 & 0 & M^2 \\ 0 & 1 & 0 & 2M & 2A \\ 0 & 0 & 1 & 0 & 3M \end{vmatrix}.$$

Subtracting the 4th row multiplied by M from the first, the determinant reduces to

$$\begin{vmatrix} 0 & A & B & C + 3MA \\ 1 & M & A & B + 2M^2 \\ -1 & 0 & 0 & M^2 \\ 0 & 1 & 0 & 3M \end{vmatrix}.$$

Again, subtracting the 2[d] column multiplied by $3M$ from the last, and reducing, the determinant becomes

$$\begin{vmatrix} 0, & B, & C \\ 1, & A, & B - M^3 \\ -1, & 0, & M^3 \end{vmatrix} = AC - B^2,$$

where M disappears, as it ought to do, because Δ is a Principiant.

In what follows we shall have frequent occasion to make use of the fact that if R_a is an absolute pure reciprocant, $\dfrac{dR_a}{a^{\frac{1}{3}}dx}$, which we know is a pure reciprocant, is also an absolute one.

This is very easily proved. For let R be any pure reciprocant, of degree i and weight w, which becomes R_a when made absolute by division by a power of a, then

$$R_a = \frac{R}{a^{\frac{\mu}{3}}}, \text{ where } \mu = 3i + w,$$

and, using G as usual to denote the generator for pure reciprocants,

$$\frac{dR_a}{dx} = \frac{GR}{a^{\frac{\mu}{3}+1}}.$$

Hence

$$\frac{dR_a}{a^{\frac{1}{3}}dx} = \frac{GR}{a^{\frac{\mu+4}{3}}},$$

which is an absolute pure reciprocant because GR, which is of degree $i+1$ and weight $w+1$, must be divided by $a^{\frac{\mu+4}{3}}$ in order to make it absolute. Thus, if M_a, A_a, B_a, C_a, ... are what M, A, B, C, ... become when each of them is made absolute by division by a power of a, we have

$$a^{-\frac{1}{3}} \frac{d}{dx} M_a = 5A_a,$$

$$a^{-\frac{1}{3}} \frac{d}{dx} A_a = 6B_a,$$

$$a^{-\frac{1}{3}} \frac{d}{dx} B_a = 7C_a + M_a A_a,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

We shall use these results in deducing the complete primitive of the differential equation

$$AC - B^2 = 0$$

from that of the equation in pure reciprocants,

$$25A^3 - 16M^3 = 0.$$

This equation may be written in the form

$$25A_a^2 = 16M_a^3;$$

whence, by differentiation, we obtain

$$50A_a\left(a^{-\frac{1}{3}}\frac{d}{dx}A_a\right) = 48M_a^2\left(a^{-\frac{1}{3}}\frac{d}{dx}M_a\right),$$

which gives $$50A_a \cdot 6B_a = 48M_a^2 \cdot 5A_a;$$

i. e. $$5B_a = 4M_a^2.$$

Differentiating this result, we find

$$5(7C_a + M_aA_a) = 40M_aA_a;$$

which gives $$C_a = M_aA_a.$$

We now restore the non-absolute reciprocants M, A, B, C; *i. e.* we write

$$5B = 4M^2 \text{ and } C = MA.$$

Hence $25(AC - B^2) = M(25A^2 - 16M^3) = 0$ (because $25A^2 = 16M^3$).

Now, the equation $AC - B^2 = 0$ remains unaltered by any homographic substitution, so that it will be satisfied not only by any solution of the equation in pure reciprocants $25A^2 - 16M^3 = 0$, but also by any homographic transformation of such solution. But it has been shown (in Lecture XIII, *American Journal of Mathematics*, Vol. IX, p. 16) that the complete primitive of $25A^2 - 16M^3 = 0$ is a linear transformation of $y = x^\lambda$, where $\lambda^2 - \lambda + 1 = 0$ (*i. e.* where λ is a cube root of negative unity).

Consequently any homographic transformation of $y = x^\lambda$ is a solution of

$$AC - B^2 = 0.$$

Moreover, this is its complete primitive; for the highest letter, f, which occurs in $AC - B^2$, corresponds to the seventh order of differentiation, and if we write

$$y = \frac{Y}{Z}, \quad x = \frac{X}{Z},$$

where X, Y, Z are general linear functions of x, y, 1 (*i. e.* if we make the most general homographic substitution), $y = x^\lambda$ becomes $Y = X^\lambda Z^{1-\lambda}$, which will be found to contain exactly 7 independent arbitrary constants. Thus the complete primitive of $AC - B^2 = 0$ is $Y = X^\lambda Z^{1-\lambda}$, where X, Y, Z are general linear functions of x, y, 1, and λ is a cube root of negative unity.

Observe that although any solution of $M = 0$ also makes A, B, C, ... all vanish, and so satisfies $AC - B^2 = 0$, we cannot from this infer that a homographic transformation of the parabola $y = x^2$ will be the complete primitive of

so that the final equation becomes

$$2^4.7^3(\lambda+1)^2(\lambda-2)^2(2\lambda-1)^2(AC-B^2)^3=3^3.5^3(\lambda^2-\lambda+1)^3A^6.$$

The same reasoning as before will show that, for a general value of λ, the complete primitive of this equation is the general homographic transformation $Y=X^\lambda Z^{1-\lambda}$ of the curve $y=x^\lambda$.

There is, however, a special exceptional case in which the differential equation becomes $$2^6.7^3(AC-B^2)^3=3^3.5^3A^6,$$

the corresponding value of the parameter λ being either 0, 1 or ∞, as may be seen by solving the equation

$$(\lambda+1)^2(\lambda-2)^2(2\lambda-1)^2=4(\lambda^2-\lambda+1)^3.$$

In the case where $\lambda=0$ or ∞ we can, in the same manner as before, show that the complete primitive is a homographic transformation of the curve $y=e^x$ by deducing the differential equation from the corresponding equation in pure reciprocants, $$25A^3+8M^3=0,$$

whose complete primitive is (see Lecture XIII) a linear transformation of $y=e^x$.

When $\lambda=1$ the corresponding equation in pure reciprocants is

$$25A^3-64M^3=0,$$

whose complete primitive may be shown to be a linear transformation of $y=x\log x$. The reason why these two distinct equations in pure reciprocants lead to the same equation in principiants is that the two curves $y=e^x$ and $y=x\log x$ are *homographically* equivalent but not *linearly* transformable into one another. For we may write the equation $y=x\log x$ in the form $x=e^{\frac{y}{x}}$, which is a homographic transformation of $y=e^x$.

Besides the special case just considered, in which the complete primitive of the equation in Principiants is $\frac{Y}{Z}=e^{\frac{X}{Z}}$, we may notice that in which the parameter λ is either -1, 2, or $\frac{1}{2}$, the differential equation reducing to $A=0$ simply, and its complete primitive $Y=X^\lambda Z^{1-\lambda}$ being the equation to a conic, as it should be. The case where $\lambda^2-\lambda+1=0$ and the differential equation reduces to $AC-B^2=0$ has been considered already. There remains the case in which $\lambda=3$, when the complete primitive becomes $YZ^2=X^3$ (the equation of the general cuspidal cubic) and the differential equation assumes the simple form $$\left(\frac{AC-B^2}{3}\right)^3=\left(\frac{A}{2}\right)^6,$$

which is therefore the differential equation of cuspidal cubics.

We shall hereafter show that in this case the Principiant

$$2^8(AC-B^2)^3-3^3A^8,$$

which is apparently of the 24^{th} degree, loses a factor a^4 and so sinks to the 20^{th} degree. It is, however, generally difficult to determine the power of a contained as a factor in a Principiant given in terms of the large letters.

The results obtained in the present Lecture agree with those of M. Halphen contained in his Thèse sur les Invariants différentiels (Paris, Gauthier-Villars, 1878), which contains a complete investigation of the properties of the Principiant $AC-B^2$, which he calls Δ. But our point of view is different from his. He obtains Δ in the form of a determinant from geometrical considerations. With him $\Delta=0$ is the differential equation which expresses the condition that, at a point x, y on any curve, a nodal cubic shall exist, having its node at x, y, and such that *one of its branches* shall have 8-point contact with the curve at that point. With us $AC-B^2$ is the simplest example, after the Mongian A, of an invariant in the capital letters A, B, C, ...

LECTURE XXXI.

We may include λ among the arbitrary constants in the primitive equation $Y=X^\lambda Z^{1-\lambda}$, which can also be written in the form

$$\lambda\log X-\log Y+(1-\lambda)\log Z=0,$$

or (X, Y, Z being general linear function of x, y, 1) in the equivalent form

$$\lambda\log(y+\alpha x+\beta)-\log(y+\alpha' x+\beta')+(1-\lambda)\log(y+\alpha'' x+\beta'')=\text{const.},$$

which evidently contains 8 independent arbitrary constants.

One of these will be made to disappear by differentiation, and thus we shall obtain a differential equation of the first order, containing 7 arbitrary constants, identical (when the constants are rearranged) with

$$(y-xt)(lx+my)+t(l'x+m'y+n')+l''x+m''y+n''=0,$$

which is known as Jacobi's Equation.

For, by differentiating the primitive equation, we obtain

$$\lambda(t+\alpha)(y+\alpha x+\beta)^{-1}-(t+\alpha')(y+\alpha' x+\beta')^{-1}$$
$$+(1-\lambda)(t+\alpha'')(y+\alpha'' x+\beta'')^{-1}=0,$$

which, when cleared of negative indices by multiplication, becomes

$$\lambda(y+\alpha'x+\beta')\{(y+\alpha''x+\beta'')(t+\alpha)-(y+\alpha x+\beta)(t+\alpha'')\}$$
$$+(y+\alpha x+\beta)\{(y+\alpha'x+\beta')(t+\alpha'')-(y+\alpha''x+\beta'')(t+\alpha')\}=0.$$

Writing this equation in the equivalent form

$$\lambda(y+\alpha'x+\beta')\{(\alpha-\alpha'')(y-xt)+(\beta''-\beta)t+(\alpha\beta''-\alpha''\beta)\}$$
$$+(y+\alpha x+\beta)\{(\alpha''-\alpha')(y-xt)+(\beta'-\beta'')t+(\alpha''\beta'-\alpha'\beta'')\}=0,$$

it is easily seen to be identical with Jacobi's equation given above.

The seven arbitrary constants which occur in Jacobi's equation are the mutual ratios of the eight coefficients l, m, l', m', n', l'', m'', n'', any one of which may have an arbitrarily chosen value assigned to it.

Taking $m=-1$, the equation may be written in the form

$$Pt+lxy-y^2+l''x+m''y+n''=0,$$

where $$P=l'x+m'y+n'-lx^2+xy.$$

In order to eliminate n'' and l'', we differentiate the above equation twice. The first differentiation gives

$$2aP+t(P'+lx-2y+m'')+ly+l''=0,$$

where $P'=\dfrac{dP}{dx}=l'+m't-2lx+y+xt$, and the second differentiation gives

$$6bP+2a(2P'+lx-2y+m'')+t(P''+2l-2t)=0.$$

Now, $P''=\dfrac{dP'}{dx}=2a(m'+x)+2(t-l)$; so that, on substituting this value, the above equation becomes

$$3bP+aQ=0, \tag{1}$$

where $$Q=2P'+lx-2y+m''+m't+xt$$
$$=2l'+3m't-3lx+3xt+m''.$$

Differentiating (1) we have

$$12cP+3bP'+3bQ+aQ'=0,$$

where $Q'=3(t-l)+6a(x+m')=3R+6aS$, suppose.

Thus we have $$4cP+bP'+bQ+aR+2a^2S=0. \tag{2}$$

Differentiating this 4 times in succession, and at each step substituting for

$$P'',\quad Q',\quad R',\quad S',$$

their values $$2R+2aS,\quad 3R+6aS,\quad 2a,\quad 1,$$

we obtain 4 more equations, from which, combined with the 2 previously obtained, we can eliminate

$$P,\ P',\ Q,\ R,\ S.$$

Thus, differentiating (2), we find

$$20dP + 8cP' + b(2R + 2aS) + 4cQ + b(3R + 6aS) + 3bR + 2a^2 + 12abS + 2a^2 = 0;$$

i. e.

$$5dP + 2cP' + cQ + 2bR + 5abS + a^2 = 0, \tag{3}$$

and continuing the same process,

$$6eP + 3dP' + dQ + 3cR + (6ac + 3b^2)S + 3ab = 0, \tag{4}$$

$$7fP + 4eP' + eQ + 4dR + (7ad + 7bc)S + (4ac + 2b^2) = 0, \tag{5}$$

$$8gP + 5fP' + fQ + 5eR + (8ae + 8bd + 4c^2)S + (5ad + 5bc) = 0. \tag{6}$$

The result of elimination is

$$\begin{vmatrix} 3b & 0 & a & 0 & 0 & 0 \\ 4c & b & b & a & 2a^2 & 0 \\ 5d & 2c & c & 2b & 5ab & a^2 \\ 6e & 3d & d & 3c & 6ac + 3b^2 & 3ab \\ 7f & 4e & e & 4d & 7ad + 7bc & 4ac + 2b^2 \\ 8g & 5f & f & 5e & 8ae + 8bd + 4c^2 & 5ad + 5bc \end{vmatrix} = 0,$$

where the determinant equated to zero is a Principiant.

In his Thèse sur les Invariants différentiels, p. 42, M. Halphen states that this equation can be found by eliminating the constants from Jacobi's equation, but he does not set out the work. When in the above determinant twice the 3[d] column is added to the second, it becomes exactly identical with the one given by Halphen, which he calls T.

We proceed to express the above result in terms of the capital letters, using the method explained in Lecture XXIX, and observing that the determinant is of degree 8 and of weight 12; so that in this case $i - w = 8 - 12 = -4$, showing that the final result has to be multiplied by a^{-4}.

Substituting in the determinant for

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ 1 & 0 & 0 & A & B & C & D + \frac{25}{8}A^2, \end{array}$$

it becomes

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 5A & 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & A & 0 & 0 & 0 \\ 7C & 4B & B & 4A & 7A & 0 \\ 8D + 25A^2 & 5C & C & 5B & 8B & 5A \end{vmatrix}.$$

Subtracting the last column multiplied by $5A$ from the first, and the 4^{th} column multiplied by 2 from the 5^{th}, and then striking out rows and columns, we obtain

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & A & 0 & 0 & 0 \\ 7C & 4B & B & 4A & -A & 0 \\ 8D & 5C & C & 5B & -2B & 5A \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & 0 & 0 & 0 \\ 7C & 4B & 4A & -A & 0 \\ 8D & 5C & 5B & -2B & 5A \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 6B & 3A & 0 & 0 \\ 7C & 4B & -A & 0 \\ 8D & 5C & -2B & 5A \end{vmatrix} = \begin{vmatrix} 6B & 3A & 0 \\ 7C & 4B & A \\ 8D & 5C & 2B \end{vmatrix}$$

$$= 24(A^2D - 3ABC + 2B^3).$$

If, using Halphen's notation, we call the principiant now under consideration T, what we have proved is that

$$T = 24a^{-4}(A^2D - 3ABC + 2B^3),$$

and consequently that $A^2D - 3ABC + 2B^3$ is divisible by a^4.

The differential equation $T = 0$ corresponds, as we have seen, to the complete primitive $Y = X^{\lambda}Z^{1-\lambda}$, in which λ is counted as one of the arbitrary constants.

This result may be otherwise obtained. For we have shown in the preceding Lecture that the differential equation of the seventh order, from which all the arbitrary constants except λ have disappeared, has the form

$$(AC - B^2)^3 = \varkappa A^8,$$

where $\varkappa$ depends solely on λ.

Writing this equation in the form

$$(AC - B^2)A^{-\frac{8}{3}} = \text{const.},$$

and differentiating with respect to x, we remove the remaining arbitrary con-

stant, and thus obtain the differential equation of the 8^{th} order free from all arbitrary constants, a result which, to a factor près, must coincide with

$$T=0.$$

We proceed to show how this differentiation may be performed without introducing any of the small letters. In the first place, it is clear that since

$$G=4(ac-b^2)\partial_b+5(ad-bc)\partial_c+6(ae-bd)\partial_d+\ldots.$$

does not contain ∂_a and is linear in the other differential reciprocals $\partial_b, \partial_c, \ldots,$

$$\begin{aligned} Ga^\theta\Phi(A, B, C, \ldots) &= a^\theta G\Phi(A, B, C, \ldots) \\ &= a^\theta\left(\frac{d\Phi}{dA}GA+\frac{d\Phi}{dB}GB+\frac{d\Phi}{dC}GC+\ldots\right). \end{aligned}$$

And since we have

$$\begin{aligned} GA &= 6B, \\ GB &= 7C+MA, \\ GC &= 8D+2MB, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

it follows immediately that

$$\begin{aligned} Ga^\theta\Phi(A, B, C, \ldots) &= a^\theta(6B\partial_A+7C\partial_B+8D\partial_C+\ldots)\Phi \\ &+ a^\theta M(A\partial_B+2B\partial_C+3C\partial_D+\ldots)\Phi. \end{aligned}$$

This is true for any function of the capital letters, whatever its nature may be; but when Φ is a principiant, it is also an invariant in the large letters; so that in this case we have

$$(A\partial_B+2B\partial_C+3C\partial_D+\ldots)\Phi=0$$

and

$$Ga^\theta\Phi=a^\theta(6B\partial_A+7C\partial_B+8D\partial_C+\ldots)\Phi.$$

Now, the operation of G on a function of degree i and weight w is equivalent to that of $a\dfrac{d}{dx}-(3i+w)b$, or to that of $a\dfrac{d}{dx}$, when both $i=0$ and $w=0$ (which happens in the case of a plenarily absolute form). Hence, if we suppose Φ to be a plenarily absolute principiant, $G\Phi$ is also a principiant, though not a plenarily absolute one.

For a is a principiant, and $\dfrac{d\Phi}{dx}$ is a principiant; therefore $a\dfrac{d\Phi}{dx}$ or $G\Phi$ is one also.* Thus

$$6B\partial_A+7C\partial_B+8D\partial_C+\ldots,$$

*See the concluding paragraph of Lecture XXV, where it was shown that P, being a principiant (of degree i and weight w), $a\dfrac{dP}{dx}-(3i+w)bP$ is a reciprocant, and $a\dfrac{dP}{dx}-(3i+2w)bP$ an invariant. This proves, what we omitted to mention there, that P being a *zero-weight* principiant,

$$GP=\left(a\frac{d}{dx}-3ib\right)P \text{ is a principiant.}$$

It may here be remarked that a principiant of degree i and of *zero weight* is equal to the corresponding plenarily absolute principiant (which is a function of the large letters only) multiplied by the factor a^i, on which the operator G does not act.

acting on any plenarily absolute principiant, generates another principiant, but not a plenarily absolute one.

We now resume the consideration of the equation

$$(AC - B^2)A^{-\frac{8}{3}} = \text{const.}$$

Differentiating and multiplying by a, we have

$$a\frac{d}{dx}\left\{(AC - B^2)A^{-\frac{8}{3}}\right\} = 0.$$

Hence, by what precedes,

$$(6B\partial_A + 7C\partial_B + 8D\partial_C)\{(AC - B^2)A^{-\frac{8}{3}}\} = 0;$$

or, using Θ to denote the operator,

$$6B\partial_A + 7C\partial_B + 8D\partial_C + \ldots.,$$

$$A^{-\frac{8}{3}}\Theta(AC - B^2) - \frac{8}{3}A^{-\frac{11}{3}}(AC - B^2)\Theta A = 0;$$

or, observing that $\Theta A = 6B$,

$$A\Theta(AC - B^2) - 16B(AC - B^2) = 0.$$

This gives $\quad A(6BC - 14BC + 8AD) - 16B(AC - B^2) = 0;$

or finally $\quad A^2D - 3ABC + 2B^3 = 0.$

We may find a generator for principiants expressed in terms of the large letters similar to the expression for the reciprocant generator G in terms of the small letters. For let P be any principiant, of weight w, which, when reduced to zero weight by division by $A^{\frac{w}{3}}$, becomes $PA^{-\frac{w}{3}}$; then

$$\Theta\left(PA^{-\frac{w}{3}}\right)$$

is a principiant. But

$$\Theta\left(PA^{-\frac{w}{3}}\right) = A^{-\frac{w}{3}-1}(A\Theta - 2wB)P,$$

where, remembering that $A^{-\frac{w}{3}-1}$ is a principiant, $(A\Theta - 2wB)P$ is one also.

Now, the weights of $\quad A, B, C, D, \ldots.$

being $\quad 3, 4, 5, 6, \ldots.,$

we may write

$$w = 3A\partial_A + 4B\partial_B + 5C\partial_C + 6D\partial_D + \ldots.,$$

and consequently

$$\begin{aligned} A\Theta - 2wB &= A(6B\partial_A + 7C\partial_B + 8D\partial_C + 9E\partial_D + \ldots.) \\ &\quad - 2B(3A\partial_A + 4B\partial_B + 5C\partial_C + 6D\partial_D + \ldots.) \\ &= (7AC - 8B^2)\partial_B + (8AD - 10BC)\partial_C + (9AE - 12BD)\partial_D + \ldots., \end{aligned}$$

which is the generator in question.

As an easy example of its use, suppose it to operate on $AC - B^2$; then

$$
\begin{aligned}
&\{(7AC - 8B^2)\,\partial_B + (8AD - 10BC)\,\partial_C\}(AC - B^2) \\
&\quad = -2B(7AC - 8B^2) + A(8AD - 10BC) \\
&\quad = 8\,(A^2D - 3ABC + 2B^3)\,.
\end{aligned}
$$

The generator just obtained,

$$(7AC - 8B^2)\,\partial_B + (8AD - 10BC)\,\partial_C + (9AE - 12BD)\,\partial_D + \ldots\ldots,$$

is a linear combination of Cayley's two generators (given in Lecture IV, Vol. VIII, p. 222 of this Journal), which, when we write A, B, C, ... instead of the corresponding small letters, become

$$(AC - B^2)\,\partial_B + (AD - BC)\,\partial_C + (AE - BD)\,\partial_D + \ldots\ldots$$

and $$(AC - 2B^2)\,\partial_B + (2AD - 4BC)\,\partial_C + (3AE - 6BD)\,\partial_D + \ldots\ldots$$

Thus we shall obtain the principiant generator by adding the second of Cayley's generators to six times the first. Either of Cayley's generators acting on a principiant would of course give an invariant in the large letters (*i. e.* a principiant), but the combination we have used has special relation to the theory of the generation of principiants by differentiation.

LECTURE XXXII.

I will now pass on to the consideration of the Principiant which, when equated to zero, gives the Differential Equation to the most general Algebraic Curve of any order.

The Differential Equation to a Conic (see the reference given on p. 18, Vol. IX of this Journal) was obtained by Monge in the first decade of this century. This was followed by the determination, in 1868, by Mr. Samuel Roberts, of the Differential Equation to the general Cubic (see Vol. X, p. 47 of Mathematical Questions and Solutions from the Educational Times). I do not consider that any substantial advance was made upon this by Mr. Muir, in the Philosophical Magazine for February, 1886, except that he sets out explicitly the quantities to be eliminated in obtaining the final result. These may of course be collected from the processes indicated by Mr. Roberts, but are not set forth by him. In speaking of the history of this part of the subject, I pass over M. Halphen's

process for obtaining the Differential Equation to a Conic. It is very ingenious, like everything that proceeds from his pen, but, being founded on the solution of a quadratic equation, does not admit of being extended to forms of a higher degree, and consequently, viewed in the light of subsequent experience, must be regarded as faulty in point of method.

Let the Differential Equation to a curve of any order, when written in its simplest form, containing no extraneous factor, be $\chi = 0$. It is convenient to give χ a single name; I call it the Criterion. The integral of the Criterion to a curve of order n must contain as many arbitrary constants as there are ratios between the coefficients of a curve of the n^{th} order. The number of these ratios being $\frac{n^2+3n+2}{2} - 1$, the order of the Criterion ought to be $\frac{n^2+3n}{2}$.

It must be independent of Perspective Projection, because projection does not affect the order of a curve. Hence it is a Principiant, and as such ought not (when y is regarded as the dependent and x as the independent variable) to contain either x, y or $\frac{dy}{dx}$ (see Lecture XXIV, *American Journal of Mathematics*, Vol. IX, p. 155).

Let $U = 0$ be an algebraical equation of the n^{th} order between x, y. I write symbolically
$$U = (p + qx + y)^n = u^n,$$
where the different powers and products of p, q, 1 which occur in the expansion of u^n are considered as representing the different coefficients in U; so that, *ex. gr.*, if $n = 3$ the coefficients of
$$y^3,\ 3y^2x,\ 3y^2,\ 3yx^2,\ 6yx,\ 3y,\ x^3,\ 3x^2,\ 3x,\ 1$$
are represented by
$$1,\ q,\ p,\ q^2,\ pq,\ p^2,\ q^3,\ pq^2,\ p^2q,\ p^3.$$
The number of terms in U is
$$1 + 2 + 3 + \ldots + (n+1) = \frac{(n+1)(n+2)}{2}.$$
The number of these containing y is
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

To obtain the Differential Equation we equate to zero the Differential Derivatives of U of all orders from $n+1$ to $\frac{n^2+3n}{2}$ inclusive, and from the $\frac{n^2+n}{2}$ equations thus formed eliminate the $\frac{n^2+n}{2}$ coefficients of the terms in U containing y.

and, in general, $u_i = y_i$ when i is any positive integer greater than 1. Thus

$$\text{co. } h^r \text{ in } \left(u + u_1 h + y_2 \frac{h^2}{1.2} + y_3 \frac{h^3}{1.2.3} + \ldots\right)^n = 0;$$

or, employing the usual modified derivatives $a, b, c, \ldots$,

$$\text{co. } h^r \text{ in } (u + u_1 h + ah^2 + bh^3 + ch^4 + \ldots)^n = 0.$$

Writing now $\qquad Q = ah^2 + bh^3 + ch^4 + \ldots,$

and expanding $(u + u_1 h + Q)^n$ in ascending powers of Q, we have

$$\text{co. } h^r \text{ in } \left\{(u + u_1 h)^n + n(u + u_1 h)^{n-1} Q + \frac{n(n-1)}{1.2}(u + u_1 h)^{n-2} Q^2 + \ldots\right\} = 0,$$

where, remembering that $r > n$, the value of co. h^r in $(u + u_1 h)^n$ is zero; so that, omitting this term, we may write

$$\text{co. } h^r \text{ in } \left\{n(u + u_1 h)^{n-1} Q + \frac{n(n-1)}{1.2}(u + u_1 h)^{n-2} Q^2 + \ldots + Q^n\right\} = 0.$$

The quantities to be eliminated will now be combinations of the various powers of u, u_1 and 1. Their number will be the same as that of the terms in $(u, u_1, 1)^{n-1}$, which is $\frac{n^2+n}{2}$, the same number as that of the equations between which the elimination is to be performed.

We now use $(m.\mu)$ to denote the coefficient of h^m in Q^μ (which, since

$$Q = ah^2 + bh^3 + ch^4 + \ldots,$$

will be independent of the combinations of u and u_1 to be eliminated), and in writing out the $\frac{n^2+n}{2}$ equations which result from making the coefficients of $h^{n+1}, h^{n+2}, \ldots h^{\frac{n^2+3n}{2}}$ in

$$n(u + u_1 h)^{n-1} Q + \frac{n(n-1)}{1.2}(u + u_1 h)^{n-2} Q^2 + \ldots + Q^n$$

vanish, we arrange their terms according to ascending values of m and μ. Thus, making the coefficient of h^{n+1} vanish, we find

$$nu_1^{n-1}(2.1) + n(n-1)u_1^{n-2}u(3.1) + \frac{n(n-1)}{1.2}u_1^{n-2}(3.2) + \ldots + (n+1.n) = 0,$$

and similarly, making the coefficient of h^{n+2} vanish,

$$nu_1^{n-1}(3.1) + n(n-1)u_1^{n-2}u(4.1) + \frac{n(n-1)}{1.2}u_1^{n-2}(4.2) + \ldots + (n+2.n) = 0.$$

So in general the equation obtained by making the coefficient of $h^{n+\kappa}$ vanish consists of a series of numerical multiples (which are independent of the value

and

$$(ah^2+bh^3+ch^4+\ldots.)^3 = a^3h^6+3a^2bh^7+(3a^2c+3ab^2)h^8 + (3a^2d+6abc+b^3)h^9+\ldots.,$$

the Cubic Criterion may be written in the form

$$\begin{vmatrix} a & b & 0 & c & a^2 & 0 \\ b & c & a^2 & d & 2ab & 0 \\ c & d & 2ab & e & 2ac+b^2 & a^3 \\ d & e & 2ac+b^2 & f & 2ad+2bc & 3a^2b \\ e & f & 2ad+2bc & g & 2ae+2bd+c^2 & 3a^2c+3ab^2 \\ f & g & 2ae+2bd+c^2 & h & 2af+2be+2cd & 3a^2d+6abc+b^3 \end{vmatrix}$$

in which it was originally obtained by Mr. Roberts.

M. Halphen has remarked that the minor of h in the Cubic Criterion is the Principiant which he calls Δ (our $AC-B^2$) multiplied by a (see p. 50 of his Thèse).

We proceed to determine the degree and weight of the Criterion of the curve of the n^{th} order. These are the same as the degree and weight of its diagonal

$$(2.1)(4.1)(5.2)(7.1)(8.2)(9.3)(11.1)(12.2)(13.3)(14.4)\ldots.,$$

which consists of $\frac{n^2+n}{2}$ factors, separable into n groups,

$$(2.1),\ (4.1)(5.2),\ (7.1)(8.2)(9.3),\ (11.1)(12.2)(13.3)(14.4),\ldots.$$

containing 1, 2, 3, 4, ... n factors respectively. Now,

$$(m.\mu) = \text{co. } h^m \text{ in } (ah^2+bh^3+ch^4+\ldots.)^\mu$$
$$= \text{co. } h^{m-2\mu} \text{ in } (a+bh+ch^2+\ldots.)^\mu,$$

and consequently $(m.\mu)$ is of degree μ and weight $m-2\mu$. Hence the degree of the Criterion (found by adding together the second numbers of the duads which occur in the diagonal) is

$$1+(1+2)+(1+2+3)+(1+2+3+4)+\ldots.+(1+2+3+\ldots.+n)$$
$$=1+3+6+10+\ldots.+\frac{n^2+n}{2}$$
$$=\frac{n(n+1)(n+2)}{6}.$$

To find the weight of the Criterion, we begin by arranging the factors of its diagonal according to their weight. This is done by writing each group of factors in reverse order, so that the diagonal is written thus:

$$(2.1)(5.2)(4.1)(9.3)(8.2)(7.1)(14.4)(13.3)(12.2)(11.1)\ldots.$$

The weights of the factors are now seen to be $0, 1, 2, 3, \ldots. \frac{n^2+n}{2}-1$; there being $\frac{n^2+n}{2}$ factors in the diagonal, one of them of zero weight. Hence the weight of the Criterion is

$$1+2+3+\ldots.+\left(\frac{n^2+n}{2}-1\right)$$
$$=\frac{\left(\frac{n^2+n}{2}-1\right)\frac{n^2+n}{2}}{2}=\frac{(n-1)n(n+1)(n+2)}{8}.$$

If, in the above formulae, we make $n=2$, we shall find that the degree is 4 and the weight 3, whereas the Mongian $a^2d-3abc+2b^3$ (which is the Criterion of the second order) is of degree 3 and weight 3.

To account for this discrepancy, observe that in this case

$$\begin{vmatrix} (2.1) & (3.1) & (3.2) \\ (3.1) & (4.1) & (4.2) \\ (4.1) & (5.1) & (5.2) \end{vmatrix} = \begin{vmatrix} a & b & 0 \\ b & c & a^2 \\ c & d & 2ab \end{vmatrix},$$

which is divisible by a, the other factor being the Mongian, as may easily be verified. This is the only case in which the determinant expression for the Criterion contains an irrelevant factor.

To express the Cubic Criterion in terms of a, A, B, C, D, E, we first remark that its degree is $\frac{3.4.5}{6}=10$, and its weight $\frac{2.3.4.5}{8}=15$. Thus the Cubic Criterion is expressible as the product of $a^{-5}(\overset{\text{deg.}}{10}-\overset{\text{wt.}}{15}=-5)$ into a function of the capital letters, which we determine by the usual method of substituting for

$$\begin{matrix} a, & b, & c, & d, & e, & f, & g & , & h & , \\ 1, & 0, & 0, & A, & B, & C, & D+\frac{25}{8}A^2, & & E+\frac{15}{2}AB. & \end{matrix}$$

When these substitutions are made, the Cubic Criterion becomes

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & A & 0 & 0 \\ 0 & A & 0 & B & 0 & 1 \\ A & B & 0 & C & 2A & 0 \\ B & C & 2A & D+\frac{25}{8}A^2 & 2B & 0 \\ C & D+\frac{25}{8}A^2 & 2B & E+\frac{15}{2}AB & 2C & 3A \end{vmatrix}.$$

Subtracting the first column of this determinant from the fifth and reducing, we obtain

$$\begin{vmatrix} 0 & 1 & A & 0 & 0 \\ A & 0 & B & 0 & 1 \\ B & 0 & C & A & 0 \\ C & 2A & D+\frac{25}{8}A^2 & B & 0 \\ D+\frac{25}{8}A^2 & 2B & E+\frac{15}{2}AB & C & 3A \end{vmatrix}.$$

Again, subtracting the second column multiplied by A from the third and reducing, there results

$$-\begin{vmatrix} A & B & 0 & 1 \\ B & C & A & 0 \\ C & D+\frac{9}{8}A^2 & B & 0 \\ D+\frac{25}{8}A^2 & E+\frac{11}{2}AB & C & 3A \end{vmatrix},$$

which, after subtracting the first row multiplied by $3A$ from the last and reducing, becomes

$$\begin{vmatrix} B & C & A \\ C & D+\frac{9}{8}A^2 & B \\ D+\frac{1}{8}A^2 & E+\frac{5}{2}AB & C \end{vmatrix}$$

$$= B\left(CD+\frac{9}{8}A^2C - BE - \frac{5}{2}AB^2\right) + C\left(BD+\frac{1}{8}A^2B - C^2\right)$$
$$+ A\left(CE+\frac{5}{2}ABC - D^2 - \frac{5}{4}A^2D - \frac{9}{64}A^4\right)$$
$$= \left(ACE - B^2E - AD^2 + 2BCD - C^3\right) - \frac{5}{4}A\left(A^2D - 3ABC + 2B^3\right) - \frac{9}{64}A^5.$$

This expression, which is of degree-weight 15.15, instead of 10.15, must be divided by a^5 to give the correct value of the Cubic Criterion.

(*To be concluded in a subsequent number.*)

Sur une Classe de Nombres remarquables.

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1. Les nombres que nous avons en vue, comme les coefficients du binôme, comme les nombres de Bernoulli et les nombres d'Euler, jouent un rôle important dans maintes formules d'Analyse. Ils ont donc été rencontrés par plusieurs auteurs parmi lesquels nous citerons MM. Schlömilch,* Catalan,† Cesaro.‡ Mais ils ont toujours été définis par ces auteurs au moyen de certaines formules d'Analyse dans lesquelles ils intervenaient. Nous les avons nous même rencontrés au cours de l'une de nos recherches.§ Aussi avons-nous pensé qu'en raison de l'importance de leur rôle, il y avait intérêt à en faire une étude directe, en partant d'une définition aussi simple que possible. C'est une telle étude que nous allons présenter ici. On y trouvera un certain nombre de remarques nouvelles.

Nous conviendrons d'abord de diverses notations qui doivent revenir fréquemment au cours de ce travail.

A_m^n représente le *produit de n nombres entiers consécutifs dont le plus grand est m*. C'est, comme on sait, le nombre des arrangements de m objets pris n à n. On convient, en outre, de poser $A_m^0 = 1$. Nous représenterons aussi, suivant l'usage, A_p^p par $p!$.

C_m^n représente le *quotient du produit de n nombres consécutifs dont le plus grand est m par le produit des n premiers nombres.* C'est le nombre des combinaisons de m objets pris n à n. On convient aussi de poser $C_m^0 = 1$.

* *Recherches sur les Coefficients des Facultés analytiques* (*Journal de Crelle*, t. 44, 1852, p. 344).

† *Sur une Suite de Polynomes entiers* . . . . (*Association française pour l'Avancement des Sciences*, t. IX, 1880, p. 78).

‡ *Sur une Équation aux Différences mêlées* (*Nouvelles Annales de Mathématiques*, 3e Série, t. IV, 1885, p. 36); *Dérivées des Fonctions de Fonctions* (*ibid.*, p. 41); *Notes sur le Calcul isobarique* (*ibid.*, p. 59).

§ *Sur un Algorithme algébrique* (*Nouvelles Annales de Mathématiques*, 3e Série, t. II, 1883, p. 220).

Nous nous bornons à ces quelques citations bibliographiques. Il serait possible assurément de les multiplier, mais les moyens nous font défaut pour le faire à l'heure où nous écrivons.

S_m^n représente la *somme des produits des m premiers nombres pris n à n.* Nous poserons encore $S_m^0 = 1$.

2. *Définition des Nombres K_m^p.*—Nous définirons les nombres que nous avons en vue au moyen d'un triangle arithmétique analogue à celui que Pascal a imaginé pour définir les coefficients du binôme.

Ce nouveau triangle est le suivant

K	1	2	3	4	5	6	7	...
1	1							
2	1	1						
3	1	3	1					
4	1	7	6	1				
5	1	15	25	10	1			
6	1	31	90	65	15	1		
7	1	63	301	350	140	21	1	
...	...	...	...	...	...	...	...	...

dont la loi de formation s'énonce ainsi :

Le bord vertical et le bord incliné de ce triangle arithmétique sont entièrement composés de 1. *Chaque nombre du triangle est égal à la somme de celui qui est placé immédiatement au-dessus de lui multiplié par le numéro de la colonne dans laquelle ils se trouvent tous deux, et de celui qui est situé immédiatement à gauche de celui-ci.*

Nous représenterons le nombre placé à l'intersection de la $m^{\text{ième}}$ ligne et de la $p^{\text{ième}}$ colonne par la notation K_m^p, de sorte que

$$K_5^2 = 15, \; K_6^3 = 90, \; K_7^5 = 140, \text{ etc.} \ldots .$$

Dans ces conditions, la loi de formation qui vient d'être énoncée tient tout entière dans les formules

$$(1) \qquad \left\{ \begin{array}{ll} K_m^1 = 1, & K_m^m = 1, \\ K_m^p = pK_{m-1}^p + K_{m-1}^{p-1}. & \end{array} \right.$$

On peut compléter le tableau précédent en mettant des 0 dans toutes les cases vides, ce qui revient à poser

$$K_m^p = 0, \text{ lorsque } p > m.$$

Nous tiendrons compte de cette convention dans la suite.

3. Certaines conséquences découlent immédiatement de la formule (1).

Par exemple, on a

$$\begin{aligned}
K_m^p &= pK_{m-1}^p + K_{m-1}^{p-1},\\
K_{m-1}^{p-1} &= (p-1)\, K_{m-2}^{p-1} + K_{m-2}^{p-2},\\
&\dots\dots\dots\dots\dots\dots\\
K_{m-p+2}^2 &= 2K_{m-p+1}^2 + K_{m-p+1}^1,\\
K_{m-p+1}^1 &= K_{m-p}^1,
\end{aligned}$$

et, en faisant la somme,

$$K_m^p = pK_{m-1}^p + (p-1)\, K_{m-2}^{p-1} + \dots + 2K_{m-p+1}^2 + K_{m-p}^1. \tag{2}$$

On a aussi

$$\begin{aligned}
K_m^p &= pK_{m-1}^p + K_{m-1}^{p-1},\\
K_{m-1}^p &= pK_{m-2}^p + K_{m-2}^{p-1},\\
&\dots\dots\dots\dots\dots\dots\\
K_{p+1}^p &= pK_p^p + K_p^{p-1},\\
K_p^p &= K_{p-1}^{p-1}.
\end{aligned}$$

Multipliant la première de ces égalités par 1, la seconde par p, la troisième par $p^2, \dots$, la dernière par p^{m-p}, et faisant la somme on a

$$K_m^p = K_{m-1}^{p-1} + pK_{m-2}^{p-1} + p^2K_{m-3}^{p-1} + \dots + p^{m-p}K_{p-1}^{p-1}. \tag{3}$$

Ce ne sont pas les seules propriétés qui puissent se déduire directement des formules de définition. En voici un autre exemple :*

Considérons le déterminant

$$\Delta_m = \begin{vmatrix}
K_1^1 & K_2^2 & K_3^3 & \dots & K_{m-1}^{m-1} & K_m^m\\
K_2^1 & K_3^2 & K_4^3 & \dots & K_m^{m-1} & K_{m+1}^m\\
K_3^1 & K_4^2 & K_5^3 & \dots & K_{m+1}^{m-1} & K_{m+2}^m\\
\dots & \dots & \dots & \dots & \dots & \dots\\
\dots & \dots & \dots & \dots & \dots & \dots\\
\dots & \dots & \dots & \dots & \dots & \dots\\
K_{m-1}^1 & K_m^2 & K_{m+1}^3 & \dots & K_{2m-3}^{m-1} & K_{2m-2}^m\\
K_m^1 & K_{m+1}^2 & K_{m+2}^3 & \dots & K_{2m-2}^{m-1} & K_{2m-1}^m
\end{vmatrix}.$$

Laissant la première colonne intacte, remplaçons les éléments de chacune des autres par leur différence avec les éléments correspondants de la colonne précé-

* Voir aussi plus loin au No. 10.

dente. Nous obtenons ainsi, en tenant compte des formules (1),

$$\Delta_m = \begin{vmatrix} K_1^1 & 0 & 0 & \ldots\ldots & 0 & 0 \\ K_2^1 & 2K_2^2 & 3K_3^3 & \ldots\ldots & (m-1)K_{m-1}^{m-1} & mK_m^m \\ K_3^1 & 2K_3^2 & 3K_4^3 & \ldots\ldots & (m-1)K_m^{m-1} & mK_{m+1}^m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{m-1}^1 & 2K_{m-1}^2 & 3K_m^3 & \ldots\ldots & (m-1)K_{2m-4}^{m-1} & mK_{2m-3}^m \\ K_m^1 & 2K_m^2 & 3K_{m+1}^3 & \ldots\ldots & (m-1)K_{2m-3}^{m-1} & mK_{2m-2}^m \end{vmatrix},$$

ou

$$\Delta_m = 2.3.4\ldots\ldots m \begin{vmatrix} K_2^2 & K_3^3 & \ldots\ldots & K_{m-1}^{m-1} & K_m^m \\ K_3^2 & K_4^3 & \ldots\ldots & K_m^{m-1} & K_{m+1}^m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{m-1}^2 & K_m^3 & \ldots\ldots & K_{2m-4}^{m-1} & K_{2m-3}^m \\ K_m^2 & K_{m+1}^3 & \ldots\ldots & K_{2m-3}^{m-1} & K_{2m-2}^m \end{vmatrix}.$$

Continuant à opérer toujours par le même procédé, on arrive, de proche en proche, à ce résultat:

(4) $$\Delta_m = 2.3^2.4^3\ldots\ldots m^{m-1}.$$

4. Les formules (1) vont nous permettre encore de démontrer cette intéressante proposition:

L'équation

$$\phi_m(x) = K_m^1 + K_m^2 x + K_m^3 x^2 + \ldots\ldots + K_m^m x^{m-1} = 0$$

obtenue en multipliant les nombres de la $m^{\text{ième}}$ *ligne par* $1,\ x,\ x^2,\ \ldots\ldots,$ *faisant la somme et égalant à* 0, A TOUTES SES RACINES RÉELLES.

La proposition est évidente pour $\phi_1(x) = 0$; il suffit donc de faire voir que si elle est vraie pour $\phi_{m-1}(x) = 0$, elle l'est encore pour $\phi_m(x) = 0$. Or, si l'équation $\phi_{m-1}(x) = 0$ a toutes ses racines réelles, il en est de même de l'équation $\psi_{m-1}(x) = x\phi_{m-1}(x) = 0$, et, par suite aussi, en vertu d'un théorème connu qui découle immédiatement du théorème de Rolle, de l'équation

$$\psi_{m-1}(x) + \psi'_{m-1}(x) = 0.$$

Mais, les formules (1) montrent que

$$\psi_{m-1}(x) + \psi'_{m-1}(x) = \phi_m(x).$$

La proposition énoncée est donc démontrée.

5. Pour avoir l'expression *explicite* du nombre K_m^p en fonction de ses indices m et p, nous ferons usage de certaines formules démontrées dans notre Note *sur un Algorithme algébrique.*

Représentant par $[a_1 a_2 \ldots . a_p]^{(m)}$ ce que devient le développement de $(a_1 + a_2 + \ldots . + a_p)^m$ lorsqu'on y remplace tous les coefficients par l'unité, nous avons démontré que si l'on pose.

$$f(x) = (x - a_1)(x - a_2) \ldots . (x - a_p),$$

on a

$$[a_1 a_2 \ldots . a_p]^{(m-p)} = \sum_{i=1}^{i=p} \frac{a_i^{m-1}}{f'(a_i)},$$

en convenant d'ailleurs de prendre

$$[a_1 a_2 . \ldots . a_p]^{(m-p)} = \begin{cases} 1 \text{ lorsque } m - p = 0 \\ 0 \quad \text{"} \quad m - p < 0 \end{cases}.$$

Or, on a, bien évidemment,

$$f'(a_i) = (a_i - a_1)(a_i - a_2) \ldots . (a_i - a_{i-1})(a_i - a_{i+1}) \ldots . (a_i - a_p);$$

donc

$$[a_1 a_2 \ldots . a_p]^{(m-p)} = \sum_{i=1}^{i=p} \frac{a_i^{m-1}}{\Pi(a_i - a_j)_{j=1,2,\ldots .,i-1,i+1,\ldots .,p}}.$$

Mais, il résulte, toujours de la même Note, que *

$$K_m^p = [123 \ldots . p]^{(m-p)}.$$

On a donc, en faisant dans la formule précédente $a_1 = 1$, $a_2 = 2, \ldots .,$ $a_p = p$,

$$K_m^p = \frac{1^{m-1}}{(1-2)(1-3)\ldots .(1-p)} + \frac{2^{m-1}}{(2-1)(2-3)\ldots .(2-p)} + \ldots .$$
$$+ \frac{i^{m-1}}{(i-1)(i-2)\ldots . 1.(-1)\ldots .(i-p)} + \ldots . + \frac{p^{m-1}}{(p-1)(p-2)\ldots . 2.1}.$$

Multipliant le premier terme haut et bas par 1, le second par 2, le troisième par 3, ... et retranscrivant la formule en intervertissant l'ordre des termes, on obtient

$$(5) \quad K_m^p = \frac{p^m}{p!} - \frac{(p-1)^m}{(p-1)!\,1!} + \frac{(p-2)^m}{(p-2)!\,2!} - \ldots .$$
$$+ (-1)^{p-2} \frac{2^m}{2!\,(p-2)!} + (-1)^{p-1} \frac{1^m}{1!\,(p-1)!}.$$

Cette formule se transforme immédiatement en celle-ci

$$(5') \quad K_m^p = \frac{p^m - C_p^1 (p-1)^m + C_p^2 (p-2)^m - \ldots . + (-1)^{p-2} C_p^{p-2} 2^m + (-1)^{p-1} C_p^{p-1} 1^m}{p!}.$$

Telle est la formule qui donne explicitement K_m^p en fonction de m et de p.

* Ce résultat est d'ailleurs immédiat. Il est bien évident, en effet, d'une part, que

$$[123 \ldots . p]^{(m-p)} = p\,[123 \ldots . p]^{(m-p-1)} + [123 \ldots . (p-1)]^{(m-p-1)};$$

de l'autre, que $\quad [1]^{m-1} = 1, \quad [123 \ldots . m]^{(0)} = 1.$

Elle a déjà été obtenue par M. Schlömilch,* mais par une méthode totalement différente et qui paraîtra sans doute moins simple. Après la convention qui a été faite sur les valeurs de $[a_1a_2 \ldots . a_p]^{(m-p)}$ pour $m-p=0$, on voit que la formule (5), ou sa transformée (5′), subsiste quelles que soient les valeurs relatives de m et de p, moyennant que l'on prenne, comme il en a déjà été convenu plus haut (No. 2), $K_p^m=0$ pour $p>m$.† Pour $m=p$, $K_m^p=1$, et la formule (5′) donne alors une remarquable identité.

6. Une autre propriété de la fonction $[a_1a_2 \ldots . a_p]^{(m-p)}$ fait connaître très-aisément la *fonction génératrice* du nombre K_m^p. Nous avons, en effet, démontré, toujours dans la même Note que $[a_1a_2 \ldots . a_p]^{(m-p)}$ est le coefficient du terme en $\frac{1}{x^m}$ dans le développement de $\frac{1}{(x-a_1)(x-a_2)\ldots .(x-a_p)}$ suivant les puissances ascendantes de $\frac{1}{x}$. Ainsi

$$\frac{1}{(x-1)(x-2)\ldots .(x-p)}=\frac{K_p^p}{x^p}+\frac{K_{p+1}^p}{x^{p+1}}+\ldots .+\frac{K_{p+m}^p}{x^{p+m}}+\ldots . \tag{6}$$

Dans ce développement en série le rapport d'un terme au précédent est

$$\frac{K_{m+1}^p}{K_m^p}\,\frac{1}{x}$$

ou, d'après la formule (5′)

$$\frac{1}{x}\cdot\frac{p^{m+1}-C_p^1(p-1)^{m+1}+C_p^2(p-2)^{m+1}-\ldots .}{p^m-C_p^1(p-1)^m+C_p^2(p-2)^m-\ldots .}$$

qui peut s'écrire

$$\frac{1}{x}\,\frac{p-C_p^1p\left(1-\frac{1}{p}\right)^{m+1}+C_p^2p\left(1-\frac{2}{p}\right)^{m+1}-\ldots .}{1-C_p^1\left(1-\frac{1}{p}\right)^m+C_p^2\left(1-\frac{2}{p}\right)^m-\ldots .}$$

dont, la limite, lorsque m croît indéfiniment est

$$\frac{p}{x}.$$

La série sera donc convergente pour

$$x>p.$$

Changeant x en $\frac{1}{x}$, ce qui transforme la condition de convergence en

$$x<\frac{1}{p},$$

* *Loc. cit.*, § 1. Ce que M. Schlömilch désigne par C_k^{-n} s'écrit, avec nos notations, K_{n+k}^n; ce qu'il désigne par C_k^n est, avec nos notations, S_{n-1}^k.

† Voir au No. 15 une autre démonstration de la formule (5′).

on obtient sous cette condition le développement

$$(7)\quad \frac{1}{(1-x)(1-2x)\ldots\ldots(1-px)} = K_p^p + K_{p+1}^p x + K_{p+2}^p x^2 + \ldots\ldots + K_m^p x^{m-p} + \ldots\ldots$$

On tire de là

$$(8)\qquad K_m^p = \frac{1}{(m-p)!}\left[D^{m-p}\frac{1}{(1-x)(1-2x)\ldots\ldots(1-px)}\right]_{x=0}.$$

7. Dans notre Note *sur un Algorithme algébrique* le nombre que nous désignons ici par K_m^p est désigné par $[1_p]^{(m-p)}$. Le nombre analogue

$$[\alpha(\alpha+1)\ldots\ldots(\alpha+p-1)]^{(m+p)},$$

désigné abréviativement par $[\alpha_p]^{(m-p)}$, pourra de même être figuré par la notation $K_m^p(\alpha)$, et on aura

$$K_m^1(\alpha) = \alpha^{m-1}, \qquad K_m^m(\alpha) = 1,$$
$$K_m^p(\alpha) = (p+\alpha-1)\,K_{m-1}^p(\alpha) + K_{m-1}^{p-1}(\alpha).$$

Par suite, les nombres $K_m^p(\alpha)$ seront donnés par le triangle arithmétique

$K(\alpha)$	1	2	3	4	
1	1				
2	α	1			
3	α^2	$2\alpha+1$	1		
4	α^3	$3\alpha^2+3\alpha+1$	$3\alpha+3$	1	
	....	....	....	....	$\ddots$

dans lequel *chaque élément de la $p^{ième}$ colonne est égal à la somme de celui qui est situé immédiatement au-dessus de lui multiplié par $p+\alpha-1$, et de celui qui est immédiatement à gauche de celui-ci.*

Les nombres $K_m^p(\alpha)$ qui généralisent les nombres K_m^p jouissent de propriétés tout-à-fait analogues. En particulier, on a

$$(9)\quad \frac{1}{(x-\alpha)(x-\alpha-1)\ldots(x-\alpha-p+1)} = \frac{K_p^p(\alpha)}{x^p} + \frac{K_{p+1}^p(\alpha)}{x^{p+1}} + \ldots + \frac{K_{p+m}^p(\alpha)}{x^{p+m}} + \ldots$$

Nous avons fait connaître (*loc. cit.*) l'expression des nombres $[\alpha_p]^{(m)}$ au moyen des nombres $[1_p]^{(m)}$. Traduisant cette formule à l'aide des notations du présent

Mémoire, après y avoir remplacé p par $m+p$, on obtient ce résultat remarquable

$$K_m^p(\alpha)=\sum_{i=0}^{i=\alpha-1} C_{\alpha-1}^i A_{p+i-1}^i K_m^{p+i}. \tag{10}$$

Il va sans dire, puisque $K_m^p=0$ pour $p>m$, que la formule précédente s'arrêtera au terme en K_m^m lorsque $p+\alpha-1$ sera supérieur à m. Ainsi

$$\begin{aligned} K_4^2(\alpha) &= K_4^2+(\alpha-1).2.K_4^3+\frac{(\alpha-1)(\alpha-2)}{1.2}.3.2.K_4^4 \\ &= 7+(\alpha-1).2.6+(\alpha-1)(\alpha-2).3. \\ &= 3\alpha^2+3\alpha+1. \end{aligned}$$

8. La formule (10) va nous permettre d'établir la condition de convergence de la série (9). En effet, dans cette série, le rapport d'un terme au précédent étant

$$\rho_m=\frac{1}{x}\cdot\frac{K_{m+1}^p(\alpha)}{K_m^p(\alpha)},$$

les formules (10) et (5′) permettront de le mettre sous la forme

$$\frac{1}{x}\cdot\frac{\begin{array}{l}p^{m+1}-C_p^1(p-1)^{m+1}+\ldots.+C_{\alpha-1}^1A_p^1[(p+1)^{m+1}-C_{p+1}^1p^{m+1}+C_{p+1}^2(p-1)^{m+1}-\ldots.] \\ +C_{\alpha-1}^2A_{p+1}^2[(p+2)^{m+1}-C_{p+2}^1(p+1)^{m+1}+C_{p+2}^2p^{m+1}-C_{p+2}^3(p-1)^{m+1}+\ldots.]+\ldots.\end{array}}{\begin{array}{l}p^m-C_p^1(p-1)^m+\ldots.+C_{\alpha-1}^1A_p^1[(p+1)^m-C_{p+1}^1p^m+C_{p+1}^2(p-1)^m-\ldots.] \\ +C_{\alpha-1}^2A_{p+1}^2[(p+2)^m-C_{p+2}^1(p+1)^m+C_{p+2}^2p^m-C_{p+2}^3(p-1)^m+\ldots.]+\ldots.\end{array}}$$

Divisant haut et bas par p^m, on voit que lorsque m croît indéfiniment, on a

$$\operatorname{Lim}\rho_m=\frac{1}{x}\operatorname{Lim}\frac{p+C_{\alpha-1}^1A_p^1\left[p\left(1+\frac{1}{p}\right)^{m+1}-C_{p+1}^1p\right]+C_{\alpha-1}^2A_{p+1}^2\left[p\left(1+\frac{2}{p}\right)^{m+1}-C_{p+2}^1p\left(1+\frac{1}{p}\right)^{m+1}+C_{p+2}^2p\right]+\ldots.}{1+C_{\alpha-1}^1A_p^1\left[\left(1+\frac{1}{p}\right)^m-C_{p+1}^1\right]+C_{\alpha-1}^2A_{p+1}^2\left[\left(1+\frac{2}{p}\right)^{m+1}-C_{p+2}^1\left(1+\frac{1}{p}\right)^{m+1}+C_{p+2}^2\right]+\ldots.}$$

expression qu'on peut écrire

$$\begin{aligned}\operatorname{Lim}\rho_m &= \frac{1}{x}\operatorname{Lim}\frac{p^{m+1}+a_1(p+1)^{m+1}+a_2(p+2)^{m+1}+\ldots.+a_{\alpha-1}(p+\alpha-1)^{m+1}}{p^m+a_1(p+1)^m+a_2(p+2)^m+\ldots.+a_{\alpha-1}(p+\alpha-1)^m} \\ &= \frac{1}{x}\operatorname{Lim}\frac{(p+\alpha-1)\left(1-\frac{\alpha-1}{p+\alpha-1}\right)^{m+1}+a_1(p+\alpha-1)\left(1-\frac{\alpha-2}{p+\alpha-1}\right)^{m+1}+\ldots.+a_{\alpha-1}(p+\alpha-1)}{\left(1-\frac{\alpha-1}{p+\alpha-1}\right)^m+a_1\left(1-\frac{\alpha-2}{p+\alpha-1}\right)^m+\ldots.+a_{\alpha-1}} \\ &= \frac{p+\alpha-1}{x}.\end{aligned}$$

La série (9) sera donc convergente si

$$x > p + a - 1.$$

9. Si on multiplie les deux membres de l'égalité (6) par $(x-1)(x-2)\dots(x-p)$ et qu'on en effectue ensuite l'identification, on arrive à la formule générale

$$K^p_{m+p} - S^1_p K^p_{m+p-1} + S^2_p K^p_{m+p-2} - \dots + (-1)^p S^p_p K^p_m = 0. \tag{11}$$

Faisant, dans cette formule, m successivement égal à m, $m+1$, $m+2$, ... $m+p$, on obtient $p+1$ équations linéaires en S^1_p, S^2_p, ..., S^p_p. Eliminant ces p quantités entre ces $p+1$ équations, on obtient la formule

$$\begin{vmatrix} K^p_{m+p} & K^p_{m+p-1} & K^p_{m+p-2} & \dots & K^p_{m+1} & K^p_m \\ K^p_{m+p+1} & K^p_{m+p} & K^p_{m+p-1} & \dots & K^p_{m+2} & K^p_{m+1} \\ K^p_{m+p+2} & K^p_{m+p+1} & K^p_{m+p} & \dots & K^p_{m+3} & K^p_{m+2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ K^p_{m+2p-1} & K^p_{m+2p-2} & K^p_{m+2p-3} & \dots & K^p_{m+p} & K^p_{m+p-1} \\ K^p_{m+2p} & K^p_{m+2p-1} & K^p_{m+2p-2} & \dots & K^p_{m+p+1} & K^p_{m+p} \end{vmatrix} = 0. \tag{12}$$

Ce déterminant est d'une forme très-symétrique. Tous les éléments d'une parallèle quelconque à la diagonale principale sont, en effet, égaux entre eux. Les indices supérieurs sont les mêmes pour tous les éléments. Cet indice commun, qui indique que tous les nombres K^p_m entrant dans la composition du déterminant appartiennent à une même colonne du triangle arithmétique de définition, est égal au nombre des éléments de chaque ligne ou de chaque colonne diminué d'une unité; quant aux indices inférieurs ils décroissent graduellement d'une parallèle à la diagonale principale à la suivante, quand on parcourt le déterminant dans le sens de la seconde diagonale prise de gauche à droite.

10. Nous allons faire voir que les formules (1) de définition permettent de démontrer directement la formule (12).

Si nous remplaçons les éléments de chaque ligne du déterminant, à l'exception de ceux de la première par leur différence avec les éléments correspondants de la ligne immédiatement supérieure multipliés par p, nous obtenons, en vertu de la formule (1), un déterminant équivalent au premier et dans lequel les indices supérieurs des éléments des p dernières lignes sont égaux à $p-1$.

Remplaçons maintenant les éléments de chacune des $p-1$ dernières lignes par leur excès sur les éléments correspondants de la ligne immédiatement supérieure, multipliés par $p-1$. Nous obtenons alors un déterminant, encore

équivalent au premier, et dans lequel les indices supérieurs des éléments des $p-1$ dernières lignes sont égaux à $p-2$.

Continuant ainsi de proche en proche, on arrive à transformer le déterminant en un autre dans lequel les indices supérieurs des deux dernières lignes sont égaux à 1. Or, d'une manière générale,

$$K_m^1 = 1.$$

Donc, dans ce dernier déterminant, les deux dernières lignes sont uniquement composées de 1; elles sont donc semblables et le déterminant est nul, ce qu'il s'agissait de prouver.

11. On sait que si l'on pose

$$\begin{aligned}
\Delta^1(x^m) &= \quad (x+1)^m \quad - x^m, \\
\Delta^2(x^m) &= \Delta^1((x+1)^m) - \Delta^1(x^m), \\
\Delta^3(x^m) &= \Delta^2((x+1)^m) - \Delta^2(x^m), \\
&\text{etc.} \ldots\ldots\ldots\ldots
\end{aligned}$$

on a, d'une manière générale,

$$\Delta^p(x^m) = (x+p)^m - C_p^1(x+p-1)^m + C_p^2(x+p-2)^m - \ldots. \\ + (-1)^{p-1}C_p^{p-1}(x+1)^m + (-1)^p C_p^p x^m.$$

Faisant, dans cette formule $x=0$, nous avons

$$\Delta^p(0^m) = p^m - C_p^1(p-1)^m + C_p^2(p-2)^m - \ldots. + (-1)^{p-1}C_p^{p-1}1^m.$$

La comparaison de cette formule avec (5') montre que

$$\Delta^p(0^m) = p!\, K_m^p. \tag{13}$$

Cette formule est très importante puisqu'elle montre, au coefficient $p!$ près, l'identité des nombres K_m^p avec les nombres $\Delta^p(0^m)$ qui se rencontrent si fréquemment en Analyse. Tous les résultats où ceux-ci interviennent, prennent, grâce à la définition si élémentaire que nous avons donnée des nombres K_m^p au No. 2, un caractère de plus grande simplicité. Nous allons en donner quelques exemples.

12. La formule d'Herschel qui fait connaître la $m^{\text{ième}}$ dérivée d'une fonction ϕ quelconque de e^x, au moyen des dérivées $\phi'(e^x)$, $\phi''(e^x)$, ... de $\phi(e^x)$ prises par rapport à e^x devient

$$D_x^m \phi(e^x) = K_m^1 e^x \phi'(e^x) + K_m^2 e^{2x}\phi''(e^x) + \ldots. + K_m^m e^{mx}\phi^{(m)}(e^x). \tag{14}$$

La formule, remarquée par M. Cesarò,* qui donne la $m^{\text{ième}}$ différence $\Delta^m y$ d'une

* *Nouvelles Annales de Mathématiques*, 1885, p. 64.

fonction quelconque, pour la différence Δx de la variable, s'écrit

$$(15)\qquad \frac{\Delta^m y}{\Delta x^m}=\sum_{i=0}^{i=\infty}\frac{K^m_{m+i}}{A^i_{m+i}}\cdot\frac{d^{m+i}y}{dx^{m+i}}\cdot\Delta x^i .$$

Pour $y=e^x$, cette formule donne, en posant $\Delta x=z$,

$$(16)\qquad \left(\frac{e^z-1}{z}\right)^m=\sum_{i=0}^{i=\infty}\frac{K^m_{m+i}}{A^i_{m+i}}z^i .$$

Pour $y=\sin x$, $\Delta x=z$,

$$(17)\qquad 2^m\sin^m\frac{z}{2}\sin\left[x+\frac{m(z+\pi)}{2}\right]=\sum_{i=0}^{i=\infty}\frac{K^m_{m+i}}{A^i_{m+i}}\sin\left[x+\frac{(m+1)\pi}{2}\right].z^{m+i}.$$

Enfin, si on pose $y=\dfrac{1}{x}$ et $\Delta x=-xz$, la formule (15) redonne la formule (7) après changement de z en x, et de m en p.

13. C'est ici le lieu de dire quelques mots des nombres de Bernoulli et d'Euler et de faire voir leur corrélation avec les nombres K^p_m.

Certains auteurs* donnent ainsi les valeurs des premiers *Nombres de Bernoulli :*

$$B_1=\frac{1}{6},\ B_2=\frac{1}{30},\ B_3=\frac{1}{42},\ B_4=\frac{1}{30},\ B_5=\frac{5}{66},\ldots.$$

d'autres† les écrivent ainsi

$$B_1=\frac{1}{6},\ B_3=\frac{1}{30},\ B_5=\frac{1}{42},\ B_7=\frac{1}{30},\ B_9=\frac{5}{66},\ldots.$$

Nous adopterons ici les notations suivantes également connues et dont M. Cesarò, entre autres, fait usage dans sa remarquable étude *sur les Nombres de Bernoulli et d'Euler,*‡ à savoir

$$B_0=1\ ,\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42},\ B_8=-\frac{1}{30},\ldots.$$
$$B_1=\frac{1}{2},\ B_3=0\ ,\ B_5=0\ ,\ B_7=0\ ,\ B_9=0\ ,\ldots.$$

Cette manière d'écrire les nombres de Bernoulli permet de les *définir* par l'égalité *symbolique* $\qquad (B+1)^\nu-B^\nu=\nu\quad(\nu=1,2,3,4,\ldots.)$

moyennant la convention que, dans le premier membre, B^i doit être remplacé par B_i.

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* Voir notamment : J. A. SERRET, *Traité de Trigonométrie*, 5e édit., p. 260.
J. TANNERY, *Introduction à la Théorie des Fonctions d'une Variable*, p. 192.

† Voir DUHAMEL, *Eléments de Calcul infinitésimal*, 3e édit., t. II, p. 427.

‡ *Nouv. Ann. de Math.* 1886, p. 305.

De même, les *Nombres d'Euler*

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385, \ldots .$$
$$E_1 = 0,\ E_3 = 0\ \ ,\ E_5 = 0,\ E_7 = 0\ \ ,\ E_9 = 0\ \ , \ldots .$$

sont définis par l'egalité symbolique

$$(E+1)^\nu + (E-1)^\nu = 0 \quad (\nu = 1, 2, 3, 4, 5, \ldots .).$$

M. Cesarò définit encore les *Nombres ultra-bernoulliens* par l'égalité symbolique

$$(\mathfrak{B}+1)^\nu - a\mathfrak{B}^\nu = \nu \quad (\nu = 0, 1, 2, 3, 4, \ldots .),$$

et les *Nombres ultra-eulériens* par cette autre

$$(\mathfrak{E}+1)^\nu + a(\mathfrak{E}-1)^\nu = 0 \quad (\nu = 1, 2, 3, 4, 5, \ldots .)$$

avec la condition initiale $\mathfrak{E}_0 = 1$.

Tous ces nombres jouent un rôle extrêmement important en Analyse. Il suffit, pour s'en rendre compte, de lire le beau travail de M. Cesarò où l'on trouvera de très-curieuses propriétés de ces nombres. Nous nous contenterons ici d'emprunter à ce Mémoire les expressions de ces diverses sortes de nombres, en fonction des nombres K_m^p.

On a

$$(18)\quad B_m = \frac{m}{2^m-1}\left[\frac{1!\,K_{m-1}^1}{2^2} - \frac{2!\,K_{m-1}^2}{2^3} + \frac{3!\,K_{m-1}^2}{2^4} - \ldots . + \frac{(-1)^m(m-1)!\,K_{m-1}^{m-1}}{2^m}\right].$$

On trouvera plus loin (No. 29) d'autres expressions de B_m.

$$(19)\quad \begin{aligned} E_m = &-\frac{1}{4}\,[2.2!\ \ K_m^2 - 2.3!\ \ K_m^3 + \ 4!\ K_m^4] \\ &+\frac{1}{4^2}\,[2.6!\ \ K_m^6 - 2.7!\ \ K_m^7 + \ 8!\ K_m^8] \\ &-\frac{1}{4^3}\,[2.10!\ K_m^{10} - 2.11!\ K_m^{11} + 12!\ K_m^{12}] \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

(Cette formule étant prolongée jusqu'à ce qu'elle s'arrête d'elle-même, c'est-à-dire jusqu'à ce qu'on tombe sur un indice d'en-haut, plus grand que m.)

$$(20)\quad \mathfrak{B}_m = -ma\left[\frac{1!\,K_{m-1}^1}{(a-1)^2} + \frac{2!\,K_{m-1}^2}{(a-1)^3} + \ldots . + \frac{(m-1)!\,K_{m-1}^{m-1}}{(a-1)^m}\right].$$

$$(21)\quad \begin{aligned} \mathfrak{E}_m = -\cos\theta.1!\,K_m^1 + \cos\theta\cos 3\theta.2!\,K_m^2 - \ldots . \\ + (-1)^{m-1}\cos^{m-1}\theta\cos(m+1)\theta.m!\,K_m^m \end{aligned}$$

θ étant défini par $\text{tang}^2\theta = a$.

Notons encore cette formule que nous avons obtenue par transformation de la formule (8) du Mémoire cité de M. Schlömilch,

$$(22)\quad \begin{aligned} S_m^p = A_m^p\Big[&\frac{A_{m+p}^p A_{m+p+1}^0}{2p!}\,K_{2p}^p - \frac{A_{m+p-1}^{p-1} A_{m+p+1}^1}{(2p-1)!}\,K_{2p-1}^{p-1} + \ldots . \\ &+ (-1)^i\,\frac{A_{m+p-i}^{p-i} A_{m+p+1}^i}{(2p-i)!}\,K_{2p-i}^{p-i} + \ldots . + \frac{A_{m+1}^1 A_{m+p+1}^{p-1}}{(p+1)!}\,K_{p+1}^1\Big]. \end{aligned}$$

Nous nous bornerons au rappel de ces quelques exemples, et nous allons reprendre l'étude des propriétés des nombres K_m^p.

14. Représentant par Y une fonction absolument quelconque, posons

$$Y_1 = \frac{d}{dx} xY,$$
$$Y_2 = \frac{d}{dx} xY_1,$$
$$\dots\dots\dots$$
$$Y_m = \frac{d}{dx} xY_{m-1}.$$

Si nous développons le calcul pour les premières de ces fonctions, nous trouvons

$$\begin{aligned} Y_1 &= Y + xY', \\ Y_2 &= Y + 3xY' + x^2Y'', \\ Y_3 &= Y + 7xY' + 6x^2Y'' + x^3Y''', \\ Y_4 &= Y + 15xY' + 25x^2Y'' + 10x^3Y''' + x^4Y^{\text{IV}}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

Le rapprochement de ces formules et du triangle arithmétique du No. 2 met bien nettement en évidence la loi de formation des coefficients dans ces formules. On vérifie d'ailleurs immédiatement, en tenant compte de la formule (1), que la loi supposée vraie jusqu'à l'indice $m-1$, l'est encore pour l'indice m. On a donc

$$Y_m = K_{m+1}^1 Y + K_{m+1}^2 xY' + \dots + K_{m+1}^m x^{m-1} Y^{(m-1)} + K_{m+1}^{m+1} x^m Y^{(m)}. \tag{23}$$

15. Cette formule est très importante. Nous allons faire voir d'abord comment elle permet de retrouver l'expression (5') de K_m^p.

Prenons pour fonction Y

$$Y = x^p.$$

Nous avons

$$\begin{aligned} Y' &= A_p^1 x^{p-1}, \\ Y'' &= A_p^2 x^{p-2}, \\ &\dots\dots\dots\dots \\ Y^{(m)} &= A_p^m x^{p-m}; \end{aligned}$$

et

$$\begin{aligned} Y_1 &= (p+1)x^p, \\ Y_2 &= (p+1)^2 x^p, \\ &\dots\dots\dots\dots \\ Y_m &= (p+1)^m x^p.* \end{aligned}$$

* Faisant $\quad Y = a_p x^p + a_{p-1} x^{p-1} + \dots + a_1 x + a_0,$

on a $\quad Y_m = (p+1)^m a_p x^p + p^m a_{p-1} x^{p-1} + \dots + 2^m a_1 x + 1^m a_0.$

L'application successive du théorème de Rolle aux équations $Y_1 = 0$, $Y_2 = 0, \dots, Y_m = 0$ permet d'énoncer ce théorème :

Si l'équation $Y = 0$ a toutes ses racines réelles,

1° *Il en est de même de l'équation $Y_m = 0$;*

2° *Cette dernière équation a autant de racines positives et autant de racines négatives que la première ;*

3° *Toute racine k^{uple} de la première équation, si $k > m$, est racine $k - m^{\text{uple}}$ de la seconde.*

La formule (23) donne donc, dans ce cas,

$$(p+1)^m x^p = K^1_{m+1} x^p + K^2_{m+1} x . A^1_p x^{p-1} + \dots \\ + K^m_{m+1} x^{m-1} A^{m-1}_p x^{p-m+1} + K^{m+1}_{m+1} x^m A^m_p x^{p-m},$$

ou, en divisant les deux membres par x^p,

$$(p+1)^m = K^1_{m+1} + A^1_p K^2_{m+1} + \dots + A^{m-1}_p K^m_{m+1} + A^m_p K^{m+1}_{m+1}. \tag{24}$$

Remarquons en passant que, si, dans cette formule, nous remplaçons p par $\frac{x}{a}$, nous pouvons l'écrire

$$(x+a)^m = K^1_{m+1} a^m + K^2_{m+1} a^{m-1} x + K^3_{m+1} a^{m-2} x(x-a) + \dots \\ + K^{m+1}_{m+1} x(x-a) \dots (x-(m-1)a), \tag{24'}$$

formule qui présente de l'analogie avec celle du binôme de Newton.

Dans la formule (24), remplaçons m par $m-1$, et faisons successivement $p = 0, 1, 2, \dots, p-1$. Nous obtenons ainsi les p équations

$$\begin{aligned} 1^{m-1} &= K^1_m, \\ 2^{m-1} &= K^1_m + A^1_1 K^2_m, \\ 3^{m-1} &= K^1_m + A^1_2 K^2_m + A^2_2 K^3_m, \\ &\dots\dots\dots\dots\dots\dots \\ p^{m-1} &= K^1_m + A^1_{p-1} K^2_m + A^2_{p-1} K^3_m + \dots + A^{p-1}_{p-1} K^p_m. \end{aligned}$$

Ces p équations du premier degré à p inconnues vont nous permettre de calculer K^p_m. Pour obtenir l'expression de cette quantité, nous emploierons l'artifice suivant: multiplions

la première équation par $(-1)^{p-1} C^{p-1}_p . 1$,
" deuxième " " $(-1)^{p-2} C^{p-2}_p . 2$,
" troisième " " $(-1)^{p-3} C^{p-3}_p . 3$,
. . . .
" dernière " " $C^0_p . p$,

et faisons la somme des équations ainsi transformées, en remarquant qu'en vertu d'une propriété bien connue des coefficients du binôme, on a

$$C^0_p C^n_p - C^1_p C^n_{p-1} + C^2_p C^n_{p-2} - \dots + (-1)^{p-n} C^{p-n}_p C^n_n = 0$$

et, par suite, en multipliant par $n!$,

$$C^0_p A^n_p - C^1_p A^n_{p-1} + C^2_p A^n_{p-2} - \dots + (-1)^{p-n} C^{p-n}_p A^n_n = 0.$$

Cette formule permet d'exprimer un nombre quelconque K_m^p au moyen de la double suite $K_{m'}^p$, $K_{m'}^{p-1}$, ..., $K_{m'}^1$ et $K_{m''}^p$, $K_{m''}^{p-1}$, ..., $K_{m''}^1$, en supposant $m' + m'' = m + 1$. Il suffit, en effet, pour cela, de remplacer, dans la formule précédente, $m + k + 1$ par m, $i + 1$ par p, $m + 1$ par m', et $k + 1$ par m''.

18. On retrouve encore les nombres K_m^p dans d'autres suites d'opérations analogues à celle du No. 14. Afin que la confusion ne soit pas possible avec celle-ci, désignons maintenant par y une fonction quelconque de x, et posons successivement

$$y_1 = x\frac{dy}{dx},$$

$$y_2 = x\frac{dy_1}{dx},$$

$$\cdots\cdots$$

$$y_m = x\frac{dy_{m-1}}{dx}.$$

Cette seconde suite d'opérations a déjà été considérée par M. Cesarò.* Le calcul successif de y_1, y_2, y_3, ... montre, d'une façon analogue à celle du No. 14, que

$$y_m = K_m^1 xy' + K_m^2 x^2 y'' + \ldots + K_m^m x^m y^{(m)}. \tag{28}$$

Désignant par Z une fonction quelconque de x, considérons encore les fonctions définies par la suite d'opérations

$$Z_1 = xZ \quad + \frac{d}{dx}xZ,$$

$$Z_2 = xZ_1 \quad + \frac{d}{dx}xZ_1,$$

$$\cdots\cdots\cdots\cdots$$

$$Z_m = xZ_{m-1} + \frac{d}{dx}xZ_{m-1}.$$

Calculant les valeurs des premières fonctions on trouve

$$Z_1 = (x+1)Z + xZ',$$

$$Z_2 = (x^2 + 3x + 1)Z + (2x + 3)xZ' + x^2Z'',$$

$$Z_3 = (x^3 + 6x^2 + 7x + 1)Z + (3x^2 + 12x + 7)xZ' + (3x + 6)x^2Z'' + x^3Z''',$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

On se trouve donc ainsi conduit à la loi remarquable exprimée par

$$Z_m = \phi_{m+1}(x)Z + \frac{\phi'_{m+1}(x)}{1}xZ' + \frac{\phi''_{m+1}(x)}{1.2}x^2Z'' + \ldots + \frac{\phi^{(m)}_{m+1}(x)}{1.2\ldots m}x^mZ^{(m)}, \tag{28$^{\text{bis}}$}$$

* *Nouv. Ann. de Math.* 1885, p. 36.

en posant, comme au No. 4,

$$\phi_{m+1}(x) = K^1_{m+1} + K^2_{m+1}x + K^3_{m+1}x^2 + \ldots\ldots + K^{m+1}_{m+1}x^m.$$

On vérifie, en effet, en tenant compte des relations (1), que cette loi supposée vraie pour l'indice $m - 1$, l'est encore pour l'indice m.

19. Les formules (23) et (28) sont susceptibles de nombreuses applications, principalement à la recherche de certains développements de fonctions.

Supposons que la fonction $f(x)$ soit, pour certaines valeurs de la variable, développable suivant les puissances de x par la formule

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_px^p + \ldots\ldots$$

Si nous posons $f(x) = Y$ et que nous appliquions la formule (23), en remarquant, pour calculer ce que devient le second membre, que si $Y = x^p$, on a $Y_m = (p+1)^m x^p$; nous obtenons

$$\begin{aligned}(29)\quad K_{m+1}f + K^2_{m+1}xf' + \ldots\ldots + K^{m+1}_{m+1}x^m f^{(m)} \\ = a_0 . 1^m + a_1 . 2^m x + \ldots\ldots + a_p(p+1)^m x^p + \ldots\ldots\end{aligned}$$

Posant de même $f(x) = y$, appliquant la formule (28) en remarquant que $y_m = p^m x^p$ pour $y = x^p$, et divisant, au résultat, les deux membres par x, on a

$$(30)\quad K^1_m f' + K^2_m xf'' + \ldots\ldots + K^m_m x^{m-1} f^{(m)} = a_1 . 1^m + a_2 . 2^m x + \ldots\ldots + a_p p^m x^{p-1} + \ldots\ldots$$

On obtient encore ce dernier résultat en posant $f'(x) = Y$, et calculant Y_{m-1} par la formule (23).

Comme on a

$$f^{(n)}(x) = A^n_n a_n + A^n_{n+1}a_{n+1}x + \ldots\ldots + A^n_{n+p}a_{n+p}x^p + \ldots\ldots,$$

si on pose $f^{(n)}(x) = Y$ et que l'on calcule ensuite Y_m par la formule (23), cela donne

$$\begin{aligned}(31)\quad K^1_{m+1}f^{(n)} + K^2_{m+1}xf^{(n+1)} + \ldots\ldots + K^{m+1}_{m+1}x^m f^{(n+m)} \\ = A^n_n a_n 1^m + A^n_{n+1}a_{n+1}2^m x + \ldots\ldots + A^n_{n+p}a_{n+p}(p+1)^m x^p + \ldots\ldots\end{aligned}$$

Cette formule comprend les deux précédentes comme cas particuliers.

Une remarque importante se place ici. Dans le dernier développement le rapport d'un terme au précédent est égal à

$$\frac{(p+n)(p+n-1)\ldots\ldots(p+1)}{(p+n-1)(p+n-2)\ldots\ldots p} \cdot \frac{(p+1)^m}{p^m} \cdot \frac{a_{n+p}}{a_{n+p-1}}\, x$$

ou

$$\left(1 + \frac{n}{p}\right)\left(1 + \frac{1}{p}\right)^m \frac{a_{n+p}}{a_{n+p-1}}\, x$$

dont la limite, lorsque p croît indéfiniment, est égale à la limite de

$$\frac{a_{n+p}}{a_{n+p-1}}\,x.$$

Si donc, on a choisi une valeur de x telle que

$$\operatorname{Lim}\frac{a_{n+p}}{a_{n+p-1}}\,x<1,$$

de façon à rendre convergent le développement de $f(x)$, cette valeur de x rendra également convergent les développements (29), (30) et (31).

Cette remarque une fois faite, nous pouvons en toute sécurité faire des applications des formules précédentes.

20. Prenons d'abord $f(x)=e^x$. Dans ce cas, on a, quel que soit x,

$$e^x=1+\frac{x}{1}+\frac{x^2}{1.2}+\ldots.+\frac{x^p}{1.2\ldots.p}+\ldots.$$

D'ailleurs toutes les dérivées de e^x étant égales à cette fonction, les formules (29), (30) et (31) conduisent toutes au même résultat que voici

$$(32)\qquad \begin{aligned} &e^x(K_{m+1}^1+K_{m+1}^2x+\ldots.+K_{m+1}^{m+1}x^m)\\ &\qquad =1^m+\frac{2^m}{1!}x+\frac{3^m}{2!}x^2+\ldots.+\frac{(p+1)^m}{p!}x^p+\ldots.\end{aligned}$$

Si, remplaçant, dans cette formule, e^x par son développement écrit plus haut, on fait l'identification des deux membres, on tombe sur la formule (24).

Si, dans cette même formule, on fait $x=1$, et que l'on pose

$$K_{m+1}^1+K_{m+1}^2+\ldots.+K_{m+1}^{m+1}=L_{m+1},$$

on a

$$(33)\qquad eL_{m+1}=1^m+\frac{2^m}{1!}+\frac{3^m}{2!}+\ldots.+\frac{(p+1)^m}{p!}+\ldots.$$

M. Cesarò, qui a obtenu cette formule, attribue la propriété qu'elle exprime à savoir que la série du second membre est égale à un multiple entier du nombre e, à M. Dobinski.

21. Arrêtons-nous un instant aux propriétés des nombres L_m si intimement liés aux nombres K_m^p.

La formule (33) donne les identités

$$\begin{aligned} eL_m\ \ &=1^{m-1}+\frac{2^{m-1}}{1!}+\frac{3^{m-1}}{2!}+\ldots.,\\ eL_{m-1}&=1^{m-2}+\frac{2^{m-2}}{1!}+\frac{3^{m-2}}{2!}+\ldots.,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ eL_1\ \ &=1\ \ \ +\frac{1}{1!}\ \ +\frac{1}{2!}\ \ +\ldots.,\end{aligned}$$

qu'on peut écrire

$$eL_m = \frac{1^m}{1!} + \frac{2^m}{2!} + \frac{3^m}{3!} + \dots,$$
$$eL_{m-1} = \frac{1^{m-1}}{1!} + \frac{2^{m-1}}{2!} + \frac{3^{m-1}}{3!} + \dots,$$
$$\dots\dots\dots\dots\dots\dots$$
$$eL_1 = \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \dots$$

Joignons-y
$$e = 1 + \frac{1^0}{1!} + \frac{2^0}{2!} + \frac{3^0}{3!} + \dots$$

Multiplions ces $m+1$ dernières égalités respectivement par

$$C_m^0,\ C_m^1,\ \dots,\ C_m^{m-1},\ C_m^m,$$

et faisons en la somme en remarquant que

$$C_m^0 a^m + C_m^1 a^{m-1} + \dots + C_m^{m-1} a + C_m^m a^0 = (a+1)^m.$$

Il vient, en posant, par convention $L_0 = 1$,

$$e(C_m^0 L_m + C_m^1 L_{m-1} + \dots + C_m^{m-1} L_1 + C_m^m L_0) = 1 + \frac{2^m}{1!} + \frac{3^m}{2!} + \frac{4^m}{3!} + \dots,$$

ou, en comparant avec la formule (33),

$$C_m^0 L_m + C_m^1 L_{m-1} + \dots + C_m^{m-1} L_1 + C_m^m L_0 = L_{m+1}. \tag{34}$$

Cette formule permet de calculer les nombres L_m par voie récurrente en partant, par convention, de $L_0 = 1$.

Elle peut s'écrire *symboliquement*,

$$L^{m+1} = (L+1)^m. \tag{34'}$$

Sous cette forme, elle a été donnée sans démonstration par M. Cesarò.* On trouve pour $L_1, L_2, L_3, \dots$ les valeurs

$$1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140,\ \dots$$

Appliquant, en partant de la formule (34') les principes du calcul symbolique, M. Cesarò a démontré, entre autres curieux résultats, que l'on a

$$e^{e^x-1} = 1 + L_1\frac{x}{1} + L_2\frac{x^2}{1.2} + L_3\frac{x^3}{1.2.3} + \dots \tag{35}$$

Nous ferons remarquer que cette formule peut s'écrire

$$e^x = 1 + \frac{L_1}{1}\log(1+x) + \frac{L_2}{1.2}[\log(1+x)]^2 + \frac{L_3}{1.2.3}[\log(1+x)]^3 + \dots \tag{35}$$

* *Nouv. Ann. de Math.* 1885, p. 39, formule (5).

La formule (35) fait connaître la fonction génératrice des nombres L_m. On a

$$L_m = \left[\frac{d^m e^{e^x-1}}{dx^m}\right]_{x=0}.$$

22. Prenons maintenant $f(x) = \frac{1}{1-x}$.

Nous avons, pour $x < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots . + x^p + \ldots .$$

Donc, si nous posons $\frac{1}{1-x} = Y$, nous avons, pour $x < 1$, par application de la formule (29),

$$(36) \quad \frac{K^1_{m+1}}{1-x} + \frac{1!\,K^2_{m+1}x}{(1-x)^2} + \ldots . + \frac{m!\,K^{m+1}_{m+1}x^m}{(1-x)^{m+1}} = 1^m + 2^m x + \ldots . + p^m x^{p-1} + \ldots .,$$

et, par application de la formule (30),

$$(37) \quad \frac{1!\,K^1_m}{(1-x)^2} + \frac{2!\,K^2_m x}{(1-x)^3} + \ldots . + \frac{m!\,K^m_m x^{m-1}}{(1-x)^{m+1}} = 1^m + 2^m x + \ldots . + p^m x^{p-1} + \ldots .$$

Cette dernière formule a été donnée par M. Catalan.* L'identification des premiers membres des formules (36) et (37) donne

$$K^1_{m+1} = 1!\,K^1_m,$$
$$C^1_m K^1_{m+1} - 1!\,K^2_{m+1} = 1!\,C^1_{m-1}K^1_m - 2!\,K^2_m,$$
$$C^2_m K^1_{m+1} - 1!\,C^1_{m-1}K^2_{m+1} + 2!\,K^3_{m+1} = 1!\,C^2_{m-1}K^1_m - 2!\,C^1_{m-2}K^2_m + 3!\,K^3_m,$$
. . . .

d'une manière générale,

$$(38) \quad \sum_{i=0}^{i=k} (-1)^i i!\,C^{k-i}_{m-i}K^{i+1}_{m+1} = \sum_{i=0}^{i=k} (-1)^i (i+1)!\,C^{k-i}_{m-i-1}K^{i+1}_m;$$

et en outre

$$K^1_{m+1} - 1!\,K^2_{m+1} + 2!\,K^3_{m+1} - \ldots . + (-1)^m m!\,K^{m+1}_{m+1} = 0.$$

Nous avons déjà trouvé cette dernière formule ; c'est la formule (26').

Quant à la formule (33), elle donne, toujours pour les valeurs de $x < 1$,

$$(39) \quad \frac{n!\,K^1_{m+1}}{(1-x)^{n+1}} + \frac{(n+1)!\,K^2_{m+1}x}{(1-x)^{n+2}} + \ldots . + \frac{(n+m)!\,K^{m+1}_{m+1}x^m}{(1-x)^{n+m+1}}$$
$$= A^n_n 1^m + A^n_{n+1}2^m x + \ldots . + A^n_{n+1}(p+1)^m x^p + \ldots .$$

* *Association française pour l'Avancement des Sciences*, Reims, 1880, p. 73.

Nous pourrions multiplier les exemples d'application des formules (29), (30) et (31); mais nous nous en tiendrons à ce qui précède.

23. Considérons l'équation différentielle linéaire

$$(40) \quad K_{m+1}^1 y + K_{m+1}^2 (x+a) \frac{dy}{dx} + K_{m+1}^3 (x+a)^2 \frac{d^2y}{dx^2} + \ldots . \\ + K_{m+1}^{m+1} (x+a)^m \frac{d^my}{dx^m} = \phi(x).$$

Elle appartient à un type que l'on soit intégrer en amenant l'équation par le changement de variable

$$x + a = e^t$$

à n'avoir que des coefficients constants. Mais elle présente une particularité digne de remarque, à savoir que l'équation algébrique, qui résout l'équation différentielle linéaire à coefficients constants à laquelle elle conduit, a toutes ses racines égales. Nous allons mettre ce fait en évidence. Calculant les premières dérivées de y par rapport à x au moyen des dérivées par rapport à t, on trouve

$$(x+a)\frac{dy}{dx} = \frac{dy}{dt},$$

$$(x+a)^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt},$$

$$(x+a)^3 \frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

On est donc conduit à poser

$$(x+a)^m \frac{d^my}{dx^m} = \frac{d^my}{dt^m} + \lambda_1 \frac{d^{m-1}y}{dt^{m-1}} + \lambda_2 \frac{d^{m-2}y}{dt^{m-2}} + \ldots . + \lambda_{m-1}\frac{dy}{dt}.$$

Pour déterminer les coefficients $\lambda_1, \lambda_2, \ldots . , \lambda_{m-1}$, faisons

$$y = (x+a)^p.$$

Il vient alors

$$\frac{d^my}{dx^m} = p(p-1)\ldots .(p-m+1)(x+a)^{p-m},$$

$$\frac{d^ky}{dt^k} = p^k e^{pt} = p^k (x+a)^p,$$

et la formule précédente donne, après suppression du facteur $p(x+a)^p$ commun aux deux membres,

$$(p-1)(p-2)\ldots .(p-(m-1)) \\ = p^{m-1} + \lambda_1 p^{m-2} + \lambda_2 p^{m-3} + \ldots . + \lambda_{m-2} p + \lambda_{m-1}.$$

On déduit de là que $\lambda_i = (-1)^i S_{m-1}^i$.

Par suite,

$$(41)\qquad (x+a)^m \frac{d^m y}{dx^m} = \frac{d^m y}{dt^m} - S_{m-1}^1 \frac{d^{m-1}y}{dt^{m-1}} + \dots + (-1)^{m-1} S_{m-1}^{m-1} \frac{dy}{dt}.$$

Cette formule transforme l'équation (40) en celle-ci

$$\left.\begin{array}{l} K_{m+1}^1 y \\ + \left(K_{m+1}^2 - S_1^1 K_{m+1}^3 + S_2^2 K_{m+1}^4 - \dots + (-1)^{m-2} S_{m-2}^{m-2} K_{m+1}^m \right. \\ \qquad\qquad \left. + (-1)^{m-1} S_{m-1}^{m-1} K_{m+1}^{m+1}\right) \dfrac{dy}{dt} \\ + \left(K_{m+1}^3 - S_2^1 K_{m+1}^4 + S_3^2 K_{m+1}^5 - \dots + (-1)^{m-2} S_{m-1}^{m-2} K_{m+1}^{m+1}\right) \dfrac{d^2y}{dt^2} \\ + \dots\dots\dots\dots\dots\dots\dots\dots \\ + \left(K_{m+1}^m - S_{m-1}^1 K_{m+1}^{m+1}\right) \dfrac{d^{m-1}y}{dt^{m-1}} \\ + K_{m+1}^{m+1} \dfrac{d^m y}{dt^m} \end{array}\right\} = \phi(e^t - a),$$

ou, en se référant à la formule (25), et posant $\phi(e^t - a) = \Phi(t)$,

$$y + C_m^1 \frac{dy}{dt} + C_m^2 \frac{d^2y}{dt^2} + \dots + C_m^{m-1} \frac{d^{m-1}y}{dt^{m-1}} + C_m^m \frac{d^m y}{dt^m} = \Phi(t).$$

Pour intégrer cette équation, on commencera par intégrer l'équation privée de second membre. L'équation résolvante de celle-ci est

$$1 + C_m^1 z + C_m^2 z^2 + \dots + C_m^{m-1} z^{m-1} + C_m^m z^m = 0$$

ou

$$(1+z)^m = 0,$$

ainsi que nous l'avions annoncé.

L'intégrale générale de l'équation privée de second membre sera donc, $c_1, c_2, \dots, c_m$ désignant des constantes arbitraires,

$$y = \frac{c_1 + c_2 t + c_3 t^2 + \dots + c_m t^{m-1}}{e^t}.$$

On en déduirait l'intégrale générale de l'équation pourvue du second membre $\Phi(t)$, en y ajoutant l'intégrale particulière de cette dernière, que permet de former la méthode de Cauchy.

Nous avons tenu à développer sur cet exemple la méthode classique, à cause de la particularité que révèle, dans ce cas, l'application de cette méthode, mais on voit tout de suite que l'équation (40) peut être intégrée par des quadratures successives.

En effet, nous n'oterons rien à la généralité de l'équation (40) en y faisant $a = 0$, et en l'écrivant

$$K_{m+1}^1 y + K_{m+1}^2 x \frac{dy}{dx} + K_{m+1}^3 x^2 \frac{d^2y}{dx^2} + \dots + K_{m+1}^{m+1} x^m \frac{d^m y}{dx^m} = \phi(x).$$

Or si, nous comparons cette équation à la formule (23), nous voyons qu'en représentant par Y l'intégrale cherchée, on a

$$Y_m = \phi(x).$$

Par suite,
$$\frac{d}{dx}\, x Y_{m-1} = \phi(x).$$

D'où, par intégration
$$x Y_{m-1} = \int \phi(x)\, dx + c_1,$$

et
$$Y_{m-1} = \frac{1}{x}\int \phi(x)\, dx + \frac{c_1}{x}.$$

De même, de proche en proche,

$$Y_{m-2} = \frac{1}{x}\int \frac{1}{x}\int \phi(x)\, dx + \frac{c_1 \log x + c_2}{x}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Y = \frac{1}{x}\int \frac{1}{x}\int \cdots \frac{1}{x}\int \phi(x)\, dx + \frac{c_1(\log x)^{m-1} + c_2(\log x)^{m-2} + \ldots + c^m}{x},$$

les signes $\int$ étant au nombre de m.

Prenons comme exemple le cas où

$$m = 2 \quad \phi(x) = \frac{1}{(1-x)^2}.$$

On tombe ainsi sur l'*Exercice No.* 520 de Frenet.* On a alors

$$\int \frac{dx}{(1-x)^2} = \frac{1}{1-x},$$

$$\frac{1}{x}\int \frac{dx}{(1-x)^2} = \frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x},$$

$$\int \frac{1}{x}\int \frac{dx}{(1-x)^2} = \int \frac{dx}{x} + \int \frac{dx}{1-x} = \log x - \log(1-x) = \log \frac{x}{1-x},$$

$$\frac{1}{x}\int \frac{1}{x}\int \frac{dx}{(1-x)^2} = \frac{1}{x}\log \frac{x}{1-x} = \log\left(\frac{x}{1-x}\right)^{\frac{1}{x}}.$$

Par suite, l'intégrale générale correspondante sera

$$\log\left(\frac{x}{1-x}\right)^{\frac{1}{x}} + \frac{c_1 \log x + c_2}{x}.$$

C'est sous cette forme que Frenet l'a obtenue par la méthode ordinaire. Nous ferons observer qu'il est plus simple de l'écrire

$$\frac{-\log(1-x) + c_1 \log x + c_2}{x}.$$

24. Nous allons maintenant faire connaître une formule, source de nombreuses identités, que nous appliquerons à plusieurs exemples, et qui nous

* *Recueil d'Exercices sur le Calcul infinitésimal*, 3e édit., pp. 244 et 315.

donnera, en particulier, diverses propriétés des nombres K^p_m. Certaines de ces propriétés, à leur tour, nous conduiront à diverses expressions des nombres de Bernoulli.

$f(n)$ représentant une fonction quelconque du nombre entier n, posons

$$\begin{aligned} f_1(n) &= f(1) + f(2) + \dots + f(n), \\ f_2(n) &= f_1(1) + f_1(2) + \dots + f_1(n), \\ &\dots\dots\dots\dots \\ f_{k+1}(n) &= f_k(1) + f_k(2) + \dots + f_k(n). \end{aligned}$$

Un calcul de proche en proche qui n'offre pas la moindre difficulté montre que

$$f_{k+1}(n) = C^k_{n+k-1} f(1) + C^k_{n+k-2} f(2) + \dots + C^k_k f(n). \tag{42}$$

Telle est la formule que nous avons en vue. Si on sait, d'une autre manière, pour une fonction $f(n)$ donnée, exprimer $f_{k+1}(n)$, cette formule fournit une identité. Nous allons en donner des exemples. L'un d'eux aura trait aux nombres K^p_m.

25. Faisons d'abord

$$f(n) = C^1_n = n.$$

Nous avons alors

$$\begin{aligned} f_1(n) &= C^1_1 + C^1_2 + \dots + C^1_n = C^2_{n+1}, \\ f_2(n) &= C^2_2 + C^2_3 + \dots + C^2_{n+1} = C^3_{n+2}, \\ &\dots\dots\dots\dots \\ f_{k+1}(n) &= C^{k+1}_{k+1} + C^{k+1}_{k+2} + \dots + C^{k+1}_{n+k} = C^{k+2}_{n+k+1}, \end{aligned}$$

et la formule (42) donne

$$C^{k+2}_{n+k+1} = C^k_{n+k-1} + 2C^k_{n+k-2} + \dots + nC^k_k. \tag{43}$$

26. Faisons maintenant

$$f(n) = a^{n-1}.$$

Nous avons alors

$$\begin{aligned} (a-1)f_1(n) &= a^n - 1, \\ (a-1)f_1(n-1) &= a^{n-1} - 1, \\ &\dots\dots\dots\dots \\ (a-1)f_1(1) &= a - 1, \end{aligned}$$

et en faisant la somme

$$(a-1)f_2(n) = a\frac{a^n-1}{a-1} - n,$$

d'où

$$\begin{aligned} (a-1)^2 f_2(n) &= a^{n+1} - a - n(a-1), \\ (a-1)^2 f_2(n-1) &= a^n - a - (n-1)(a-1), \\ &\dots\dots\dots\dots \\ (a-1)^2 f_2(1) &= a^2 - a - (a-1), \end{aligned}$$

et, en faisant encore la somme

$$(a-1)^2 f_3(n) = a^2 \frac{a^n - 1}{a-1} - na - \frac{n(n+1)}{1.2}(a-1),$$

ou $$(a-1)^3 f_3(n) = a^{n+2} - a^2 - C_n^1 a(a-1) - C_{n+1}^2 (a-1)^2.$$

En continuant ainsi de proche en proche, on arrive à la formule suivante

$$(a-1)^{k+1} f_{k+1}(n) = a^{n+k} - [a^k + C_n^1 a^{k-1}(a-1) + C_{n+1}^2 a^{k-2}(a-1)^2 + \ldots\ldots + C_{n+k-1}^k (a-1)^k].$$

La formule (42) donne donc, dans ce cas,

$$(44)\quad (a-1)^{k+1}[C_{n+k-1}^k + C_{n+k-2}^k a + \ldots\ldots + C_k^k a^{n-1}] = a^{n+k} - [a^k + C_n^1 a^{k-1}(a-1) + C_{n+1}^2 a^{k-2}(a-1)^2 + \ldots\ldots + C_{n+k-1}^k (a-1)^k].$$

Pour $a = 2$, cette formule devient

$$(45)\quad C_{n+k-1}^k + 2C_{n+k-2}^k + \ldots\ldots + 2^{n-1} C_k^k = 2^{n+k} - [2^k + 2^{k-1} C_n^1 + 2^{k-2} C_{n+1}^2 + \ldots\ldots + C_{n+k-1}^k].$$

Pour $a = -1$,

$$(46)\quad 2^{k+1}[C_{n+k-1}^k - C_{n+k-2}^k + \ldots\ldots + (-1)^{n-1} C_k^k] = (-1)^{n-1} + [1 + C_n^1 2 + C_{n+1}^2 2^2 + \ldots\ldots + C_{n+k-1}^k 2^k].$$

27. Nous allons maintenant, en faisant

$$f(n) = n^m$$

rencontrer des propriétés des nombres K_m^p.

Reprenons les formules obtenues au No. 15, en faisant $p = 0, 1, 2, 3 \ldots\ldots$ dans la formule (24) où on remplace m par $m - 1$. Ces formules peuvent s'écrire, en multipliant les deux membres de la 1ère par 1, ceux de la 2ème par 2, ..., ceux de la $n$ième par n,

$$1^m = C_1^1 K_m^1,$$
$$2^m = C_2^1 K_m^1 + 2!\ C_2^2 K_m^2,$$
$$3^m = C_3^1 K_m^1 + 2!\ C_3^2 K_m^2 + 3!\ C_3^3 K_m^3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$n^m = C_n^1 K_m^1 + 2!\ C_n^2 K_m^2 + 3!\ C_n^3 K_m^3 + \ldots\ldots + n!\ C_n^n K_m^n.$$

Faisant la somme, nous avons*

$$(47)\quad 1^m + 2^m + \ldots\ldots + n^m = f_1(n) = 1!\ C_{n+1}^2 K_m^1 + 2!\ C_{n+1}^3 K_m^2 + \ldots\ldots + n!\ C_{n+1}^{n+1} K_m^n.$$

Continuant un calcul analogue de proche en proche on trouve

$$(47')\quad f_{k+1}(n) = 1!\ C_{n+k+1}^{k+2} K_m^1 + 2!\ C_{n+k+1}^{k+3} K_m^2 + \ldots\ldots + n!\ C_{n+k+1}^{n+k+1} K_m^n.$$

*Si on avait $n > m$, le second membre s'arrêterait évidemment au terme $n!\ C_{n+1}^{m+1} K_m^m$. La formule (47) n'en est pas moins générale, puisque nous avons fait la convention de prendre $K_m^n = 0$ pour $n > m$. Même remarque pour les formules suivantes.

La formule (42) donne alors

$$(48)\quad C^k_{n+k-1}1^m + C^k_{n+k-2}2^m + \dots + C^k_k n^m = 1!\, C^{k+2}_{n+k+1}K^1_m \\ + 2!\, C^{k+3}_{n+k+1}K^2_m + \dots + n!\, C^{n+k+1}_{n+k+1}K^n_m.$$

On peut, pour $f(n) = n^m$, obtenir $f_{k+1}(n)$ d'une autre manière. Il suffit, en effet, tout simplement, de remplacer $m-1$ par m dans les formules du No. **15**, dont nous venons de parler. Cela donne

$$1^m = C^0_0 K^1_{m+1},$$
$$2^m = C^0_1 K^1_{m+1} + 1!\, C^1_1 K^2_{m+1},$$
$$3^m = C^0_2 K^1_{m+1} + 1!\, C^1_2 K^2_{m+1} + 2!\, C^2_2 K^3_{m+1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$n^m = C^0_{n-1}K^1_{m+1} + 1!\, C^1_{n-1}K^2_{m+1} + 2!\, C^2_{n-1}K^3_{m+1} + \dots + (n-1)!\, C^{n-1}_{n-1}K^n_{m+1};$$

d'où, par addition,

$$(49)\quad 1^m + 2^m + \dots + n^m = f_1(n) = C^1_n K^1_{m+1} + 1!\, C^2_n K^2_{m+1} + \dots \\ + (n-1)!\, C^n_n K^n_{m+1}.$$

De proche en proche, toujours par le même procédé, on arrive à

$$(49')\quad f_{k+1}(n) = C^{k+1}_{n+k}K^1_{m+1} + 1!\, C^{k+2}_{n+k}K^2_{m+1} + \dots + (n-1)!\, C^{n+k}_{n+k}K^n_{m+1}.$$

La formule (42) donne alors

$$(50)\quad C^k_{n+k-1}1^m + C^k_{n+k-2}2^m + \dots + C^k_k n^m \\ = C^{k+1}_{n+k}K^1_{m+1} + 1!\, C^{k+2}_{n+k}K^2_{m+1} + \dots + (n-1)!\, C^{n+k}_{n+k}K^n_{m+1}.$$

Enfin, une troisième méthode se présente pour le calcul de $f_{k+1}(n)$, en supposant toujours $f(n) = n^m$.

En effet, la formule (26) donne

$$0 = K^1_{m+1} - 1!\, C^1_1 K^2_{m+1} + 2!\, C^2_2 K^3_{m+1} - \dots + (-1)^m m!\, C^m_m K^{m+1}_{m+1},$$
$$(-1)^m = K^1_{m+1} - 1!\, C^1_2 K^2_{m+1} + 2!\, C^2_3 K^3_{m+1} - \dots + (-1)^m m!\, C^m_{m+1}K^{m+1}_{m+1}.$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-n)^m = K^1_{m+1} - 1!\, C^1_{n+1}K^2_{m+1} + 2!\, C^2_{n+2}K^3_{m+1} - \dots + (-1)^m m!\, C^m_{m+n}K^{m+1}_{m+1}.$$

En additionnant ces égalités, on a

$$(51)\quad (-1)^m[1^m + 2^m + \dots + n^m] = (-1)^m f_1(n) \\ = C^1_{n+1}K^1_{m+1} - 1!\, C^2_{n+2}K^2_{m+1} + \dots + (-1)^m m!\, C^{m+1}_{m+n+1}K^{m+1}_{m+1}.$$

Faisant successivement dans cette formule $n = 1, 2, 3, \dots, n$, plaçant en outre, en tête des égalités ainsi obtenues, la formule (26′), et additionnant, on trouve

$$(-1)^m f_2(n) = C^2_{n+2}K^1_{m+1} - 1!\, C^3_{n+3}K^2_{m+1} + \dots + (-1)^m m!\, C^{m+2}_{m+n+2}K^{m+1}_{m+1}.$$

Continuant ainsi, de proche en proche, en faisant intervenir à chaque fois la formule (26), on arrive finalement au résultat que voici

$$(51') \quad (-1)^m f_{k+1}(n) = C^{k+1}_{n+k+1} K^1_{m+1} - 1!\, C^{k+2}_{n+k+2} K^2_{m+1} + \ldots\ldots + (-1)^m m!\, C^{m+k+1}_{m+n+k+1} K^{m+1}_{m+1}.$$

Par suite, la formule (42) donne

$$(52) \quad \begin{aligned} & C^k_{n+k-1} 1^m + C^k_{n+k-2} 2^m + \ldots\ldots + C^k_k n^m \\ & = (-1)^m [C^{k+1}_{n+k+1} K^1_{m+1} - 1!\, C^{k+2}_{n+k+2} K^2_{m+1} + \ldots\ldots \\ & + (-1)^m m!\, C^{m+k+1}_{m+n+k+1} K^{m+1}_{m+1}]. \end{aligned}$$

28. Si nous faisons encore

$$f(n) = 2n - 1,$$

nous avons $\qquad f_1(n) = 1 + 3 + 5 + \ldots\ldots + 2n - 1 = n^2.$

Donc, en posant $\qquad f'(n) = n^2,$

on a $\qquad f_{k+1}(n) = f'_k(n).$

Or, $f'_k(n)$ c'est $f_k(n)$ du No. précédent en supposant $m = 2$. On a donc, d'après la formule (47′) où on a remplacé k par $k - 1$,

$$\begin{aligned} f_{k+1}(n) = f'_k(n) &= 1!\, C^{k+1}_{n+k} K^1_2 + 2!\, C^{k+2}_{n+k} \\ &= C^{k+1}_{n+k} + 2 C^{k+2}_{n+k} \\ &= C^{k+2}_{n+k+1} + C^{k+2}_{n+k}. \end{aligned}$$

La formule (42) donne donc

$$(53) \quad C^k_{n+k-1} + 3 C^k_{n+k-2} + 5 C^k_{n+k-3} + \ldots\ldots + (2n-1)\, C^k_k = C^{k+2}_{n+k+1} + C^{k+2}_{n+k}.$$

Si au lieu de la formule (47′) on prend (49′), en y remplaçant encore k par $k - 1$, on a pour le cas actuel

$$\begin{aligned} f_{k+1}(n) &= C^k_{n+k-1} K^1_3 + 1!\, C^{k+1}_{n+k-1} K^2_3 + 2!\, C^{k+2}_{n+k-1} K^3_3 \\ &= C^k_{n+k-1} + 3 C^{k+1}_{n+k-1} + 2 C^{k+2}_{n+k-1} \\ &= C^k_{n+k-1} + \quad C^{k+1}_{n+k-1} + 2 (C^{k+1}_{n+k-1} + C^{k+2}_{n+k-1}) \\ &= C^{k+1}_{n+k} \quad + 2 C^{k+2}_{n+k}. \end{aligned}$$

On retombe donc sur la formule (53).

Enfin la formule (51′), toujours après remplacement de k par $k - 1$, et pour $m = 2$, donne

$$\begin{aligned} f_{k+1}(n) &= C^k_{n+k} K^1_3 - 1!\, C^{k+1}_{n+k+1} K^2_3 + 2!\, C^{k+2}_{n+k+2} K^3_3 \\ &= C^k_{n+k} - 3 C^{k+1}_{n+k+1} + 2 C^{k+2}_{n+k+2} \\ &= 2 (C^{k+2}_{n+k+2} - C^{k+1}_{n+k+1}) - (C^{k+1}_{n+k+1} - C^k_{n+k}) \\ &= 2 C^{k+2}_{n+k+1} - C^{k+1}_{n+k} \\ &= C^{k+2}_{n+k+1} + (C^{k+2}_{n+k+1} - C^{k+1}_{n+k}) \\ &= C^{k+2}_{n+k+1} + C^{k+2}_{n+k}. \end{aligned}$$

On est ainsi encore ramené à la formule (53).

29. En adoptant la définition des nombres B_m de Bernoulli, que nous avons admise au No. 13, en vertu de laquelle

$$B_0 = 1\ ,\quad B_2 = \frac{1}{6}\,,\quad B_4 = -\frac{1}{30}\,,\quad B_6 = \frac{1}{42}\,,\ \ldots\ldots,$$
$$B_1 = \frac{1}{2}\,,\quad B_3 = 0\ ,\quad B_5 = 0\quad ,\quad B_7 = 0\ ,\ \ldots\ldots,$$

on sait que B_m est le coefficient de n dans le développement de

$$s_m = 1^m + 2^m + \ldots + n^m$$

suivant les puissances de n.

Or, la formule (47) donne

$$s_m = 1!\,K_m^1\,\frac{(n+1)n}{2!} + 2!\,K_m^2\,\frac{(n+1)n(n-1)}{3!} + 3!\,K_m^3\,\frac{(n+1)n(n-1)(n-2)}{4!} + \ldots$$

On en déduit, en prenant le coefficient de n,

$$B_m = \frac{K_m^1}{2} - \frac{1!\,K_m^2}{3} + \frac{2!\,K_m^3}{4} - \ldots + (-1)^{m-1}\frac{(m-1)!\,K_m^m}{m+1}. \tag{54}$$

La formule (49) donne à son tour

$$s_m = K_{m+1}^1\,\frac{n}{1!} + 1!\,K_{m+1}^2\,\frac{n(n-1)}{2!} + 2!\,K_{m+1}^3\,\frac{n(n-1)(n-2)}{3!} + \ldots$$

On en déduit de même

$$B_m = \frac{K_{m+1}^1}{1} - \frac{1!\,K_{m+1}^2}{2} + \frac{2!\,K_{m+1}^3}{3} - \ldots + (-1)^m\frac{m!\,K_{m+1}^{m+1}}{m+1}. \tag{55}$$

Enfin la formule (51) donne

$$s_m = (-1)^m K_{m+1}^1\,\frac{n+1}{1!} + (-1)^{m+1}1!\,K_{m+1}^2\,\frac{(n+2)(n+1)}{2!}$$
$$+ (-1)^{m+2}2!\,K_{m+1}^3\,\frac{(n+3)(n+2)(n+1)}{3!} + \ldots,$$

d'où l'on tire

$$B_m = (-1)^m K_{m+1}^1 + (-1)^{m+1}\frac{S_2^1}{2}\,K_{m+1}^2 + (-1)^{m+2}\frac{S_3^2}{3}\,K_{m+1}^3 + \ldots + \frac{S_{m+1}^m}{m+1}\,K_{m+1}^{m+1}. \tag{56}$$

Cette dernière expression est moins commode pour le calcul que les deux précédentes.

On trouvera dans une Note, qui va paraître cette année au *Bulletin de la Société mathématique de France*, d'autres applications de la formule (42).

Extraits de deux Lettres addressées à M. Craig.

PAR M. HERMITE.

SUR LA FORMULE DE FOURIER.

* * * * Je suppose qu'on ait entre les limites $x = 0$, $x = 2\pi$:

$$f(x) = \Sigma A_m e^{mix},$$

l'indice m parcourant la série des nombres entiers, $0, \pm 1, \pm 2$, etc. Décomposons maintenant cette série en deux autres, et soit:

$$\Phi(x) = \frac{1}{2} A_0 + \Sigma A_m e^{mix},$$

$$(m = 1, 2, 3, \ldots .)$$

puis:

$$\Psi(x) = \frac{1}{2} A_0 + \Sigma A_{-m} e^{-mix},$$

$$(m = 1, 2, 3, \ldots .)$$

de sorte qu'on aura: $f(x) = \Phi(x) + \Psi(x)$.

Je vais établir que dans le demi plan situé au dessus de l'axe des abscisses, c'est à dire pour toutes les valeurs imaginaires, $z = x + iy$ où y est une quantité positive différente de zéro, on a cette expression:

$$\Phi(z) = \frac{1}{4i\pi} \int_0^{2\pi} f(x) \cot \frac{x - z}{2} dx.$$

Et semblablement si l'on suppose y négatif:

$$\Psi(z) = -\frac{1}{4i\pi} \int_0^{2\pi} f(x) \cot \frac{x - z}{2} dx;$$

ce sera donc l'extension de chacune des fonctions, dans les régions considérées, qu'on obtient au moyen de $f(x)$, et en employant les seules valeurs réelles de la variable qui sont comprises entre $x = 0$ et $x = 2\pi$.

Pour cela je fais usage des relations suivantes:

$$\int_0^{2\pi} e^{mix} \cot \frac{x - z}{2} dx = 4i\pi e^{miz},$$

$$\int_0^{2\pi} \cot \frac{x - z}{2} dx = 2i\pi,$$

$$\int_0^{2\pi} e^{-mix} \cot \frac{x - z}{2} dx = 0,$$

qui ont lieu pour m positif, la variable z représentant un point dont l'ordonnée est positive. Elles font voir que dans l'intégrale $\int_0^{2\pi} f(x)\cot\frac{x-z}{2}\,dx$, les termes affectés des coefficients A_m où l'indice est négatif, disparaissent, et nous en concluons immédiatement l'expression annoncée:

$$\Phi(z)=\frac{1}{4i\pi}\int_0^{2\pi} f(x)\cot\frac{x-z}{2}\,dx.$$

On a ensuite, dans la région inférieure du plan, m étant toujours un entier positif:

$$\int_0^{2\pi} e^{mix}\cot\frac{x-z}{2}\,dx=0,$$

$$\int_0^{2\pi}\cot\frac{x-z}{2}\,dx=-2i\pi,$$

$$\int_0^{2\pi} e^{-mix}\cot\frac{x-z}{2}\,dx=-4i\pi e^{-miz},$$

et ces relations nous donnent:

$$\Psi(z)=-\frac{1}{4i\pi}\int_0^{2\pi} f(x)\cot\frac{x-z}{2}\,dx.$$

A la formule de Fourier:

$$f(x)=\Sigma A_m e^{mix},$$

je joint ainsi la fonction uniforme dans tout le plan:

$$\Phi(z)=\frac{1}{4i\pi}\int_0^{2\pi} f(x)\cot\frac{x-z}{2}\,dx$$

qui a l'axe des abscisses pour coupure, de sorte qu'en designant par N et N' deux points infiniment voisins, l'un au dessus l'autre au dessous de l'axe, on a la relation:

$$\Phi(N)-\Phi(N')=f(x).$$

Je remarquerai encore que la considération de cette coupure donne immédiatement les intégrales définies qui viennent d'être employées. Qu'on pose en effet:

$$J=\int_0^{2\pi} e^{mix}\cot\frac{x-z}{2}\,dx,$$

d'où:

$$Je^{-miz}=\int_0^{2\pi} e^{mi(x-z)}\cot\frac{x-z}{2}\,dx,$$

on trouve d'abord:

$$D_z(Je^{-miz})=0.$$

Soit donc $Je^{-miz}=C$, l'expression de cette constante par l'intégrale montre qu'elle s'évanouit pour z infiniment grand et au dessous de l'axe des abscisses; on a par conséquent $C=0$ dans le demi plan au dessous de cet axe. Franchissons la coupure, l'intégrale en passant du point N' au point N l'augmente de $4i\pi$, et dans le demi plan au dessus de la coupure, on obtient

$$Je^{-miz}=4i\pi,$$

d'où:

$$J=4i\pi e^{miz}.$$

Mais j'ai supposé l'entier m positif et différent de zéro; on trouve quand il est

nul, $\cot\frac{x-z}{2}=-i\pi$, ou $+i\pi$, pour une valeur infinie de z, au dessous puis au dessus de l'axe des abscisses, et l'on en conclut alors, $J=-2i\pi$, $J=+2i\pi$ pour chaucun des demi plans. Le cas de m négatif, se traiterait le même.

ADDITIONS.

En donnant communication à M. Lipschitz des résultats qui précédent, j'ai été informé qu'ils se trouvaient établis par une autre voie, dans son ouvrage Lehrbuch der Analysis, T. II, p. 724. La note suivante expose la méthode suivie par l'illustre géomêtre.

Soit $f(x+iy)$ une fonction uniforme et continue pour toutes les valeurs $x+iy$, où $x^2+y^2\leqq 1$, et qui prend pour $x+iy=0$ une valeur réelle, en outre soit désigné par $g(x+iy)$ la fonction, qui est conjugée à $f(x+iy)$. Alors pour chaque valeur $x+iy$ à l'intérieur du cercle $x^2+y^2<1$, on a l'expression

$$f(x+iy)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(e^{i\alpha})+g(e^{-i\alpha}))\left(\frac{1}{1-e^{-i\alpha}(x+iy)}-\frac{1}{2}\right)d\alpha.$$

En remplaçant la variable complexe $x+iy$ par la fonction exponentielle $e^{i\omega}$, la variable nouvelle ω doit avoir une partie imaginaire positive, et l'équation proposée prend la forme suivante:

$$f(e^{i\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(e^{i\alpha})+g(e^{-i\alpha}))\left(\frac{1}{1-e^{i\omega-i\alpha}}-\frac{1}{2}\right)d\alpha.$$

La démonstration est ramenée au théorème de Cauchy à l'aide de la remarque que le second facteur, qui se trouve sous le signe intégral, peut être écrit

soit $$\frac{1}{i}\left(\frac{d(e^{i\alpha})}{e^{i\alpha}-e^{i\omega}}-\frac{1}{2}\frac{d(e^{i\alpha})}{e^{i\alpha}}\right),$$

ou $$\frac{1}{i}\left(\frac{d(e^{-i\alpha})}{e^{-i\alpha}-e^{-i\omega}}-\frac{1}{2}\frac{d(e^{-i\alpha})}{e^{-i\alpha}}\right).$$

Cela étant l'intégrale proposée se trouve égale à la somme des deux intégrales

$$\frac{1}{2\pi i}\int_{-\pi}^{\pi}f(e^{i\alpha})\left(\frac{d(e^{i\alpha})}{e^{i\alpha}-e^{i\omega}}-\frac{1}{2}\frac{d(e^{i\alpha})}{e^{i\alpha}}\right),$$

et $$\frac{1}{2\pi i}\int_{-\pi}^{\pi}g(e^{-i\alpha})\left(\frac{d(e^{-i\alpha})}{e^{-i\alpha}-e^{-i\omega}}-\frac{1}{2}\frac{d(e^{-i\alpha})}{e^{-i\alpha}}\right).$$

En appliquant le théorème de Cauchy on voit facilement, que la première intégrale prend la valeur $f(e^{i\omega})-\frac{1}{2}f(0)$, la seconde intégrale la valeur $\frac{1}{2}g(0)$. A cause de la supposition, que $f(0)$ doit être une quantité réelle, la différence $-\frac{1}{2}f(0)+\frac{1}{2}g(0)$ s'évanouit. Partant, la somme des deux intégrales s'égale à

la valeur $f(e^{i\omega})$, ce qu'il fallait prouver. Si l'on fait usage de l'équation

$$\frac{1}{1-e^{i\omega-i\alpha}}-\frac{1}{2}=\frac{1}{2i}\operatorname{cotg}\left(\frac{\alpha-\omega}{2}\right),$$

le résultat en question passe dans la forme suivante:

$$f(e^{i\omega})=\frac{1}{4\pi i}\int_{-\pi}^{\pi}(f(e^{i\alpha})+g(e^{-i\alpha}))\operatorname{cotg}\left(\frac{\alpha-\omega}{2}\right)d\alpha.$$

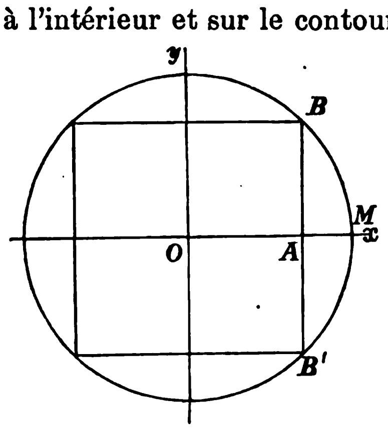
Sur une Formule de Gauss.

Dans le mémoire intitulé: De nexu inter multitudinem classium, etc. (Œuvres de Gauss, T. II, p. 269), on trouve l'expression suivante du nombre des valeurs entières de x et y qui satisfont à la condition: $x^2+y^2 \overline{\underset{<}{=}} A$.

Soit r l'entier contenu dans $\sqrt{A}$, et q l'entier contenu dans $\sqrt{\frac{1}{2}A}$; désignons aussi par $r^{(q+1)}$, $r^{(q+2)}$, ... les entiers les plus voisins de $\sqrt{A-(q+1)^2}$, $\sqrt{A-(q+2)^2}$, ..., jusqu'à $\sqrt{A-r^2}$; le nombre cherché est:

$$4q^2+1+4r+8\left[r^{(q+1)}+r^{(q+2)}+\ldots.+r^{(r)}\right].$$

Pour démontrer cette formule je remarquerai d'abord qu'on obtient facilement le nombre des points dont les coordonnées sont des nombres entiers et qui sont à l'intérieur d'un rectangle ayant ses cotés parallèles aux axes et son centre à l'origine. Nommons la base et la hauteur $2a$ et $2b$, soit ensuite p et q les entiers contenus dans a et b, le produit $(2p+1)(2q+1)$ sera le nombre du points considérés qui sont à l'intérieur et sur le contour du rectangle.



Cela posé, inscrivons un carré dans le cercle $x^2+y^2=A$, on aura

$$OA=AB=\sqrt{\frac{1}{2}A},$$

et si l'on désigne par q l'entier contenu dans $\sqrt{\frac{1}{2}A}$, le nombre des points qui

sont dans le carré et sur son contour, sera $(2q+1)^2$. Il faut maintenant y joindre ceux qui se trouvent dans les quatre segments égaux à BMB'; et dont voici l'énumération.

Sur AM nous avons en premier lieu les points dont les abscisses sont: $q+1$, $q+2, \ldots. r$, r désignant comme plus haut l'entier contenu dans $\sqrt{A}$; leur nombre est par conséquent, $r-q$.

A ces diverses abscisses correspondent les ordonnées:

$$\sqrt{A-(q+1)^2},\ \sqrt{A-(q+2)^2}, \ldots. \sqrt{A-r^2},$$

et en employant la notation de Gauss, nous avons sur la première un nombre de points égal à $r^{(q+1)}$, sur la seconde à $r^{(q+2)}$, etc.; donc dans le segment BMB', un nombre égal à:

$$r-q+2\left[r^{(q+1)}+r^{(q+2)}+\ldots.+r^{(r)}\right].$$

Quadruplons cette valeur et ajoutons à celle que nous avons obtenue pour le carré inscrit, on trouve la quantité

$$(2q+1)^2+4(r-q)+8\left[r^{(q+1)}+r^{(q+2)}+\ldots.+r^{(r)}\right],$$

qui se réduit à l'expression de Gauss:

$$4q^2+1+4r+8\left[r^{(q+1)}+r^{(q+2)}+\ldots.+r^{(r)}\right].$$

D'une manière toute semblable s'obtient le nombre des points contenus à l'intérieur et sur le contour de l'ellipse:

$$Ay^2+Bx^2=N.$$

Soit à cet effet, en désignant par $E(x)$ l'entier contenu dans x:

$$y_\xi = E\left(\sqrt{\frac{N-B\xi^2}{A}}\right),\quad x_\eta = E\left(\sqrt{\frac{N-A\eta^2}{B}}\right),$$

$$a = E\left(\sqrt{\frac{N}{A}}\right),\qquad b = E\left(\sqrt{\frac{N}{B}}\right),$$

$$\alpha = E\left(\sqrt{\frac{N}{2A}}\right),\qquad \beta = E\left(\sqrt{\frac{N}{2B}}\right),$$

nous avons cette formule dont celle de Gauss est un cas particulier:

$$4\alpha\beta+1+2(a+b)+2(x_{\alpha+1}+x_{\alpha+2}+\ldots.+x_a)$$
$$+\,2(y_{\beta+1}+y_{\beta+2}+\ldots.+y_b).$$

SUR L'EXPRESSION DU SINUS PAR UN PRODUIT DE FACTEURS PRIMAIRES.

La formule suivante qui a été donnée pour la première fois par M. Weierstrass:

$$\sin x = x\Pi\left[\left(1-\frac{x}{n\pi}\right)e^{\frac{x}{n\pi}}\right]\ (n=\pm 1, \pm 2, \ldots.),$$

conduit facilement à une expression semblable pour $\cos x$, au moyen de l'équation

$$\cos x = \frac{\sin 2x}{2 \sin x}.$$

Soit en effet, $m = \pm 1, \pm 3, \pm 5$, etc., nous pouvons écrire :

$$\sin x = x \Pi \left[\left(1 - \frac{x}{2n\pi}\right) e^{\frac{x}{2n\pi}} \right] \Pi \left[\left(1 - \frac{x}{m\pi}\right) e^{\frac{x}{m\pi}} \right];$$

tous les facteurs de $\sin x$ se trouveront ainsi mis en évidence dans $\sin 2x$, on en conclut :

$$\cos x = \Pi \left[\left(1 - \frac{2x}{m\pi}\right) e^{\frac{2x}{m\pi}} \right].$$

Mais on pourrait désirer parvenir à cette expression, en partant de la relation $\cos x = \sin\left(\frac{\pi}{2} + x\right)$, c'est ce que je vais faire au moyen d'une remarque sur la formule générale :

$$F(x) = \Pi \left[\left(1 - \frac{x}{a_n}\right) e^{P_n(x)} \right],$$

où les polynômes $P_n(x)$ sont de degrés quelconques.

Changeons x en $x + \xi$, et employons l'identité :

$$1 - \frac{x+\xi}{a_n} = \left(1 - \frac{\xi}{a_n}\right)\left(1 - \frac{x}{a_n - \xi}\right);$$

on aura d'abord :

$$F(x+\xi) = \Pi \left[\left(1 - \frac{\xi}{a_n}\right)\left(1 - \frac{x}{a_n - \xi}\right) e^{P_n(x+\xi)} \right];$$

divisons ensuite membre à membre avec l'égalité :

$$F(\xi) = \Pi \left[\left(1 - \frac{\xi}{a_n}\right) e^{P_n(\xi)} \right]$$

et nous obtiendrons la formule

$$\frac{F(x+\xi)}{F(\xi)} = \Pi \left[\left(1 - \frac{x}{a_n - \xi}\right) e^{P_n(x+\xi) - P_n(\xi)} \right].$$

D'une manière semblable, et en partant de la relation :

$$F(x) = x \Pi \left[\left(1 - \frac{x}{a_n}\right) e^{P_n(x)} \right],$$

on trouverait :

$$\frac{F(x+\xi)}{F(\xi)} = \left(1 + \frac{x}{\xi}\right) \Pi \left[\left(1 - \frac{x}{a_n - \xi}\right) e^{P_n(x+\xi) - P_n(\xi)} \right].$$

Mais ce résultat appliqué à $\sin x$, en supposant $\xi = \frac{\pi}{2}$, donne l'expression suivante :

$$\cos x = \left(1 + \frac{2x}{\pi}\right) \Pi \left[\left(1 - \frac{2x}{(2n-1)\pi}\right) e^{\frac{x}{n\pi}} \right];$$

$$(n = \pm 1, \pm 2, \text{etc.})$$

qui ne coïncide pas avec la formule obtenue tout-à-l'heure:

$$\cos x = \Pi\left[\left(1 - \frac{2x}{m\pi}\right)e^{\frac{2x}{m\pi}}\right].$$

On remarque toutefois qu'en posant $m = 2n - 1$, les facteurs exponentiels $e^{\frac{x}{n\pi}}$ et $e^{\frac{2x}{m\pi}}$, tendent vers la même limite, lorsque le nombre entier n augmente, mais la différence entre les deux résultats doit être expliquée; voici une considération qui lèvera toute difficulte!

Reprenons l'équation dont se tire l'expression de sinus par un produit de facteurs primaires:

$$\cot x = \frac{1}{x} + \sum\left[\frac{1}{x - n\pi} + \frac{1}{n\pi}\right];$$
$$(n = \pm 1, \pm 2, \ldots .),$$

et d'où on conclut en changeant x en $x + \xi$:

$$\cot(x + \xi) = \frac{1}{x + \xi} + \sum\left[\frac{1}{x + \xi - n\pi} + \frac{1}{n\pi}\right].$$

Retranchons membre à membre avec l'égalité:

$$\cot a = \frac{1}{a} + \sum\left[\frac{1}{a - n\pi} + \frac{1}{n\pi}\right],$$

où a désigne une constante arbitraire, on aura ainsi:

$$\cot(x + \xi) - \cot a = \frac{1}{x + \xi} - \frac{1}{a} + \sum\left[\frac{1}{x + \xi - n\pi} - \frac{1}{a - n\pi}\right],$$

et plus simplement:

$$\cot(x + \xi) - \cot a = \sum\left[\frac{1}{x + \xi - n\pi} - \frac{1}{a - n\pi}\right],$$

en supposant maintenant $n = 0, \pm 1, \pm 2$, etc.

De la se tire si nous intégrons depuis $x = 0$:

$$\log\frac{\sin(x + \xi)}{\sin \xi} - x \cot a = \sum\left[\log\left(1 - \frac{x}{n\pi - \xi}\right) + \frac{x}{n\pi - a}\right],$$

et par conséquent:

$$\frac{\sin(x + \xi)}{\sin \xi} = e^{x \cot a}\Pi\left[\left(1 - \frac{x}{n\pi - \xi}\right)e^{\frac{x}{n\pi - a}}\right].$$

La quantité a dans cette formule est quelconque, on peut même la prende égale à zéro. Qu'on mette à part en effet le facteur correspondant à $n = 0$ qui est seul à considérer dans ce cas $\left(1 + \frac{x}{\xi}\right)e^{-\frac{x}{a}}$; on observera que pour $a = 0$ la

différence $\cot a - \frac{1}{a}$ s'évanouit, de sorte qu'on obtient alors :

$$\frac{\sin(x+\xi)}{\sin\xi} = \left(1+\frac{x}{\xi}\right)\Pi\left[\left(1-\frac{x}{n\pi-\xi}\right)e^{\frac{x}{n\pi}}\right].$$

Ce résultat conduit en supposant $\xi = \frac{\pi}{2}$, à l'expression considérée plus haut :

$$\cos x = \left(1-\frac{2x}{\pi}\right)\Pi\left[\left(1-\frac{2x}{(2n-1)\pi}\right)e^{\frac{x}{n\pi}}\right].$$

Je change ensuite ξ en $\xi+\frac{\pi}{2}$ et a en $a+\frac{\pi}{2}$, on trouve ainsi en posant $m = 2n-1$:

$$\frac{\cos(x+\xi)}{\cos\xi} = e^{-x\,\mathrm{tg}\,a}\Pi\left[\left(1-\frac{2x}{m\pi-\xi}\right)e^{\frac{2x}{m\pi-a}}\right]$$

$$(m = \pm 1,\ \pm 3,\ \pm 5,\ \ldots),$$

d'où pour $\xi = 0$ et $a = 0$:

$$\cos x = \Pi\left[\left(1-\frac{2x}{m\pi}\right)e^{\frac{2x}{m\pi}}\right].$$

On voit donc que les deux expressions différentes que nous avions remontrées s'accordent, puisqu'elles ne sont que des cas particuliers d'une formule plus générale.

Two Proofs of Cauchy's Theorem.

BY F. FRANKLIN.

The following proofs of Cauchy's theorem that $\int_1^2 wdz$ has the same value for any two paths joining the points 1 and 2, provided that w is a uniform and continuous function of $z\ (= x + i)$ throughout the area included between those paths, are very simple; and I think they have the merit of more pointedly turning on the fundamental property that dw/dz is independent of the direction of dz than do the proofs usually given.

1°. Let the integral be taken along a certain path, and let the path be slightly deformed; denote the effect of this deformation by δ; then

$$\begin{aligned}\delta\int wdz &= \int\delta\,(wdz) = \int\delta w\,.\,dz + \int w\,.\,\delta\,(dz) = \int\delta w\,.\,dz + \int wd\,(\delta z)\\ &= [w\delta z]_1^2 + \int(\delta w\,.\,dz - \delta z\,.\,dw)\\ &= \int(\delta w\,.\,dz - \delta z\,.\,dw),\end{aligned}$$

since δz is zero at the points 1 and 2. All this is true whether w be a function of z or merely a function of x and y. But if w is a function of z, uniform and continuous in the region in question, $\delta w/\delta z = dw/dz$, or $\delta w\,.\,dz - \delta z\,.\,dw = 0$; hence the deformation does not affect the integral. This proves the theorem.

2°. Consider an infinitesimal contour, c, containing the point z_0. At the point z_0 let $w = w_0$, and put $z = z_0 + \zeta$, $w = w_0 + \omega$. Then

$$\begin{aligned}\int_c wdz &= \int_c(w_0 + \omega)\,d\zeta = w_0\int_c d\zeta + \int_c \omega d\zeta\\ &= \int_c \omega d\zeta,\end{aligned}$$

since $\int_c d\zeta$ is obviously 0. Now, if w is a function of z (uniform and continuous

in the region considered), ω/ζ is constant, $= A$ say, around the contour by the fundamental property already referred to; hence

$$\int_c w\,dz = A\int_c \zeta\, d\zeta = \frac{1}{2} A \int_c d\,(\zeta^2) = 0\,.$$

Thus the integral taken around an infinitesimal contour vanishes. Hence, by addition, the integral taken around any contour vanishes, and the theorem is proved.

Strictly speaking, what was proved about $\int_c w\,dz$ is not that it is absolutely 0, but that it is an infinitesimal of a higher order than the second; but this is of course sufficient for the purpose.

It may be added that neither of these proofs depends on the fact that $z = x + iy$; they are equally applicable if z is any function of x and y, and w a function of z; z and w being supposed uniform and continuous in the region concerned.

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